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Arrow's Impossibility Theorem: Two Simple Single-Profile Versions

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Abstract

In this paper we provide two simple new versions of Arrow's impossibility theorem, in a model with only one preference profile. Both versions are transparent, requiring minimal mathematical sophistication. The first version assumes there are only two people in society, whose preferences are being aggregated; the second version assumes two or more people. Both theorems rely on assumptions about diversity of preferences, and we explore alternative notions of diversity at some length. Our first theorem also uses a neutrality assumption, commonly used in the literature; our second theorem uses a neutrality/monotonicity assumption, which is stronger and less commonly used. We provide examples to illustrate our points.

5.1 Introduction

In 1950 Kenneth Arrow ([Ar1],[Ar2]) provided a striking answer to a basic abstract problem of democracy: how can the preferences of many individuals be aggregated into social preferences? The starkly negative answer, known as Arrow's impossibility theorem, was that every conceivable aggregation method has some flaw. That is, a handful of reasonable-looking axioms, which one thinks an aggregation procedure should satisfy, lead to impossibility: the axioms are mutually inconsistent. This impossibility theorem created a large literature and major field called social choice theory; see for example, Suzumura's ([Su]) Introduction to the *Handbook of Social Choice and Welfare*, and the Campbell and Kelly ([CK]) survey in the same volume.[1]

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[1]The theorem has also had a major impact on the larger fields of economics and political science, as well as on distant fields like mathematical biology. (See, *e.g.*, Day and McMorris ([DM]).)

In this paper we develop two very simple versions of Arrow's impossibility theorem. Our models are so-called single-profile models. This means impossibility is demonstrated in the context of one fixed profile of preferences, rather than in the (standard) Arrow context of many varying preference profiles.

Single-profile Arrow theorems were first proved in the late 1970's and early 1980's by Parks ([Pa]), Hammond ([Ha]), Kemp and Ng ([KN]), Pollak ([Po]), Roberts ([Ro]) and Rubinstein ([Ru]). Single-profile theorems were developed in response to an argument of Paul Samuelson ([Sa1]) against Arrow. Samuelson claimed that Arrow's model, with varying preference profiles, is irrelevant to the classical problem of maximizing a Bergson-Samuelson-type social welfare function (Bergson ([Be])), which depends on a given set of ordinal utility functions, that is, a fixed preference profile. The single-profile Arrow theorems established that negative results, such as dictatorship, or illogic of social preferences, or, more generally, impossibility of aggregation, could be proved with one fixed preference profile (or set of ordinal utility functions), provided the profile is "diverse" enough.

This paper has two purposes. The first is to provide two short and transparent single-profile Arrow theorems. In addition to being short and simple, our theorems do not require the existence of large numbers of alternatives. Our second purpose is to explore the meaning of preference profile diversity. Our first Arrow impossibility theorem, which is extremely easy to prove, assumes that there are only two people in society. The proof relies on a neutrality assumption and our first version of preference diversity, which we call simple diversity. In our second Arrow impossibility theorem, which is close to Pollak's ([Po]) version, there are two or more people. For this version we strengthen neutrality to neutrality/monotonicity, and we use a second, stronger version of preference diversity, which we call complex diversity.

Other recent related literature includes Geanakoplos ([Ge]), who has three very elegant proofs of Arrow's theorem in the standard multi-profile context, and Ubeda ([Ub]) who has another elegant multi-profile proof.[2] These proofs, while short, are mathematically much more challenging than ours. Reny ([Re]) has an interesting side-by-side pair of (multi-profile) proofs, of Arrow's theorem and the related theorem of Gibbard and Satterthwaite.

5.2 The Model

We assume a society with two or more individuals, and three or more alternatives. A specification of the preferences of all individuals is called a preference profile. In our theorems there is only one preference profile. The preference profile is somehow transformed into a social preference relation. This might be done through a voting process, through the actions of an enlightened government, or by the force of a dictator. Any kind of social choice process is possible in Arrow's world. The individual preference relations are all assumed to be complete and transitive. Both the individual and the social preference relations allow indifference. The following notation is used: Generic alternatives are $x, y, z, w, \ldots$. Particular alternatives are $a, b, c, d, \ldots$. A generic person is labeled $i, j, k, \ldots$; a particular person is $1, 2, 3, \ldots$. Person i's preference relation is R_i. xR_iy means person i prefers x to y or is indifferent between them; xP_iy means i prefers x to y; xI_iy means i is indifferent between them. Society's preference relation is R. xRy means society prefers x to y or is indifferent between them; xPy means society prefers x to y; xIy means society is indifferent between them. We start with the following assumptions[3]:

(1) **Complete and transitive social preferences**. The social preference relation R is complete and transitive.

(2.a) **Weak Pareto principle**. For all x and y, if xP_iy for all i, then xPy.

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[2] Ubeda also emphasizes the importance of (multi-profile) neutrality, similar to but stronger than the assumption we use in this paper, and much stronger than Arrow's independence assumption, and he provides several theorems establishing neutrality's equivalence to other intuitively appealing principles.

[3] Assumptions are just assumptions, and are not necessarily true. In fact, Arrow's problem is to show that a set of assumptions is inconsistent: if all but one are true, then the remaining one must be false.

(2.b) **Strong Pareto principle**. For all x and y, if xR_iy for all i, and xP_iy for some i, then xPy.

(3.a) **Neutrality**. Suppose individual preferences for w versus z are identical to individual preferences for x versus y. Then the social preference for w versus z must be identical to the social preference for x versus y. Formally: For all x, y, z, and w, assume that, for all i, xP_iy if and only if wP_iz and zP_iw if and only if yP_ix. Then wRz if and only if xRy, and zRw if and only if yRx.

(4) **No dictator**. There is no dictator. Individual i is a **dictator** if, for all x and y, xP_iy implies xPy.

(5.a) **Simple diversity**. There exists a triple of alternatives x, y, z such that xP_iy for all i, but opinions are split on x versus z and on y versus z. That is, some people prefer x to z and some people prefer z to x, and, similarly, some people prefer y to z and some people prefer z to y.

Note that we have two alternative versions of the Pareto principle here. The first (weak Pareto) is more common in the Arrow's theorem literature (e.g., see Campbell and Kelly ([CK, p. 42])). We will use the strong Pareto principle in our two-person impossibility theorem below, and the weak Pareto principle in our two-or-more person impossibility theorem. Neutrality, assumption (3.a), and simple diversity, assumption (5.a), are so numbered because we will introduce alternatives later.

Also note that the no dictator assumption is different in a world with a single preference profile from what it is in the multi-profile world. For example, in the single-profile world, if all individuals have the same preferences, and if Pareto holds (weak or strong), then by definition everyone is a dictator. Or, if individual i is indifferent among all the alternatives, he is by definition a dictator. We will discuss this possibility of innocuous dictatorship in Section 5.9 below.

5.3 Some Examples in a Two-Person Model

We illustrate with a few simple examples. For these there are two people and three alternatives, and we assume no individual indifference between any pair of alternatives. Given that we aren't allowing individual indifference, the two Pareto principles collapse into one. Preferences of the two people are shown by listing the alternatives from top (most preferred) to bottom (least preferred). In our examples, the last column of the table shows what is being assumed about society's preferences. The comment below each example indicates which desired property is breaking down. The point of these examples is that if we are willing to discard any one of our five basic assumptions, the remaining four may be mutually consistent.

Person 1	Person 2	Society (Majority Rule)
a	c	
b	a	aPb, aIc, & bIc
c	b	

Example 1. Transitivity for social preferences fails. Transitivity for R implies transitivity for I. This means aIc & cIb should imply aIb. But we have aPb.

Person 1	Person 2	Society
a	c	
b	a	$aIbIc$
c	b	

Example 2. Pareto (weak or strong) fails, because aP_1b and aP_2b should imply aPb. But we have aIb.

Person 1	Person 2	Society
a	c	a
b	a	c
c	b	b

Example 3. Neutrality fails. Compare the social treatment of a versus c, where the two people are split and person 1 gets his way, to the social treatment of b versus c, where the two people are split and person 2 gets his way.

Person 1	Person 2	Society (1 is Dictator)
a	c	a
b	a	b
c	b	c

Example 4. There is a dictator.

Note that Examples 1 through 4 all use the same profile of individual proferences, which satisfies the simple diversity assumption. The next example modifies the individual preferences:

Person 1	Person 2	Society (Majority Rule)
a	c	
c	a	aIc
b	b	aPb & cPb

Example 5. Simple diversity fails. Opinions are no longer split over two pairs of alternatives.

5.4 Neutrality, Independence, and Some Preliminary Arrow Paradoxes

One of the most controversial of Arrow's original assumptions was independence of irrelevant alternatives. We did not define it above because it does not play a direct role in single-profile Arrow theorems; however it lurks behind the scenes. Therefore we define it at this point. Independence requires the existence of multiple preference profiles, and to accommodate multiple profiles, we use primes: Person i's preference relation was shown as R_i above, and society's as R; at this point we will write R_i' and R' for alternative preferences for person i and society, respectively. Now consider a pair of alternatives x and y. Arrow's independence of irrelevant alternatives condition requires appropriate consistency in the social ranking of x and y as individual preferences switch from unprimed to primed. More formally:

(6) **Independence**. Let $R_1, R_2, \ldots$ and R be one set of individual and social preference relations and $R_1', R_2', \ldots$ and R' be another. Assume that for all i, xP_iy if and only if $xP_i'y$ and $yP_i'x$ if and only if yP_ix. Then xRy if and only if $xR'y$ and $yR'x$ if and only if yRx.

Note the parallel between the independence assumption and the neutrality assumption. Independence involves multiple preference profiles whereas our version of neutrality assumes there is one preference profile. Independence focuses on a pair of alternatives and switches between two preference profiles, one unprimed and the other primed. It says that if the x versus y individual preferences are the same under the two preference profiles, then the x versus y social preference must also be the same. This statement is of course meaningless if there is only one preference profile. The closest analogy when there is only one preference profile is neutrality, which says that if individual preferences regarding x versus y under the one fixed preference profile are the same as individual preferences regarding w versus z under that profile, then the x versus y social preference must be the same as the w versus z social preference.

In short, in a single-profile model, independence is a vacuous assumption, and its natural replacement is neutrality.[4]

This natural replacement, however, prompted Samuelson to launch an attack in [Sa2] directed at the Kemp's and Ng's neutrality assumption in [KN]. Samuelson called neutrality, among other things, "anything but reasonable," and "gratuitous." ([Sa2]) He offered the following *reductio ad absurdum* example:

[4]The definition of neutrality can be easily extended to a multi-profile model, and neutrality is a stronger assumption than independence in such a model.

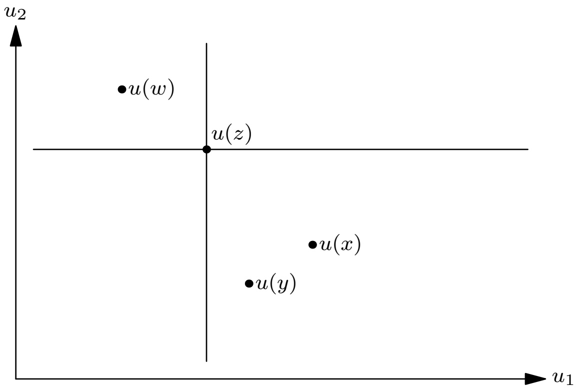

Figure 5.1: Fleurbaey and Mongin's Arrow impossibility argument.

Example 6 (Samuelson's Chocolates)**.** There are two people. There is a box of 100 (indivisible) chocolates to be distributed between them. They both like chocolates, and each is hungry enough to eat them all. The alternatives are $x_0 = (100, 0), x_1 = (99, 1), x_2 = (98, 2), \dots,$ where the first number is the number of chocolates going to person 1 and the second is the number going to person 2.

Many ethical observers, looking at this society, would say that x_1 is better than x_0. That is, $x_1 P x_0$. That is, it would be a good thing to take a chocolate from person 1, when he has 100 of them, and give it to person 2. Note that $x_0 P_1 x_1$ and $x_1 P_2 x_0$.

Now consider any $k < 100$. The individual preferences are $x_k P_1 x_{100}$ and $x_{100} P_2 x_k$, similar to the individual preferences for x_0 versus x_1. By neutrality, $x_{100} P x_k$! That is, society should give all the chocolates to person 2!

Samuelson's chocolates example is a vivid attack on neutrality, but should not be viewed as a compelling reason to drop it. One response to the example is to say society should not decide that x_1 is better than x_0 in the first place; if society simply found x_0 and x_1 equally good (contrary to the instincts of the chocolate redistributionist), neutrality would have implied that all the x's are socially indifferent. This would have been perfectly logical. Another response is to observe that neutrality is a property of extremely important and widely used decision-making procedures, particularly majority voting, and therefore cannot be lightly dismissed. In fact, any social decision procedure that simply counts instances of xP_iy, yP_ix, and xI_iy, but does not weigh strength of feelings, satisfies neutrality.

Samuelson ([Sa2]) also offered a graphical argument against Arrow's theorem with neutrality, an argument that was simplified and improved years later by Fleurbaey and Mongin ([FM]), as follows:

Fleurbaey and Mongin Graphical Arrow Impossibility Argument. Assume that there are two people, and some set of alternatives $x, y, z, \dots$. Assume the individuals have utility functions u_1 and u_2, so $u_1(x)$, for example, represents person 1's utility level from alternative x.

Consider the graph in Figure 5.1. Utility levels of individuals 1 and 2 are on the horizontal and vertical axes, respectively. Each alternative shows up in the graph as a utility pair, for instance $u(z) = (u_1(z), u_2(z))$ represents alternative z. We start at $u(z)$ and draw horizontal and vertical lines through it, creating four quadrants.

Now assume complete and transitive social preferences, strong Pareto and neutrality. Take two alternatives, say x and y, whose utility vectors are within the *southeast* quadrant. Choose them so that $u(x)$ is northeast of $u(y)$.

Social indifference between z and x is impossible, for the following reasons: First, by neutrality, if zIx, then zIy, must also hold. Second, if zIx and zIy, then xIy by transitivity. But third, since $u(x)$ is northeast of $u(y), xPy$ by Pareto.

Therefore either society prefers z to x, or society prefers x to z. Suppose xPz. Now consider another alternative w. By neutrality, if $u(w)$ is in the *northwest* quadrant (as in Figure 5.1), xPz implies zPw. By neutrality, if $u(w)$ is in the *southeast* quadrant, xPz implies wPz. By strong Pareto, if $u(w)$ is in the *northeast* quadrant, wPz. By strong Pareto, if $u(w)$ is in the *southwest* quadrant, zPw. But this argument establishes that social preferences (for w versus z) are always exactly the same as person 1's; that is, person 1 is a dictator. Had we started out by assuming zPx, person 2 would have been the dictator. In short, the graph produces an Arrow impossibility. □

There are two drawbacks to the Fleurbaey/Mongin/(Samuelson) graphical impossibility argument. First, it has the disadvantage that it requires the use of the utility functions u_1 and u_2—it is cleaner to dispense with utility functions and simply use preference relations for individuals. Second, it incorporates a crucial diversity assumption without being explicit about it. Assuming the existence of the triple of utility vectors $u(x)$, $u(y)$, and $u(z)$, with their respective locations in the utility diagram, is in fact exactly the assumption of simple diversity: both 1 and 2 prefer x to y, but opinions are split on x versus z and opinions are split on y versus z. In Theorem 8 below, we make this assumption explicit.

5.5 Arrow Impossibility Theorem, $n = 2$

We are ready to turn to our own simple version of Arrow's impossibility theorem, in the single-profile model. Throughout this section, we assume there are two people in society. We will show that our five assumptions, complete and transitive social preferences, strong Pareto, neutrality, simple diversity, and no dictator, are mutually inconsistent.

First we establish Proposition 7, which is by itself a very strong result. This proposition corresponds to the Samuelson's chocolates example, and so we call it Samuelson's chocolates proposition. Then we prove our first simple version of Arrow's theorem.

Proposition 7 (Samuelson's Chocolates)**.** *Assume $n = 2$. Assume the strong Pareto principle and neutrality. Suppose for some pair of alternatives x and y, and for the two people i and j, xP_iy and yP_jx. Suppose that xPy. Then person i is a dictator.*

Proof. Let w and z be any pair of alternatives. Assume wP_iz. We need to show that wPz must hold. If wR_jz, then wPz by strong Pareto. If not, wR_jz, then zP_jw by completeness for j's preference relation, and then wPz by neutrality. □

Theorem 8 (Arrow Impossibility Theorem)**.** *Assume $n = 2$. The assumptions of complete and transitive social preferences, strong Pareto, neutrality, simple diversity, and no dictator are mutally inconsistent.*

Proof. By simple diversity there exist x, y and z such that xP_iy for $i = 1, 2$, but such that opinions are split on x versus z, and on y versus z.

Now xPy by the Pareto principle, weak or strong. Since opinions are split on x versus z, one person prefers x to z, while the other prefers z to x. If xPz, then the person who prefers x to z is a dictator by Proposition 7. If zPx, then the person who prefers z to x is a dictator by Proposition 7.

Suppose then that xIz. Then zIx. By transitivity, zIx and xPy implies zPy. But opinions are split on y versus z. Therefore one person prefers z to y, and the other person prefers y to z. By Proposition 7, the person who prefers z to y is a dictator. We have shown that whatever the social preference for x and z might be, there must be a dictator. □

5.6 Trying to Generalize to an n-Person Model

In what follows we seek to generalize our version of Arrow's theorem to societies with two or more people. In order to get an impossibility theorem when $n \geq 2$, we need to strengthen some of our basic assumptions. We start with the neutrality assumption. We will strengthen it to a single-profile version of what is called neutrality/monotonicity.[5] The intuition is that if everybody who prefers

[5] See Blau & Deb ([BD]), who call the multi-profile analog "full neutrality and monotonicity"; Sen ([Se]), who calls it NIM; and Pollak ([Po]), who calls it "nonnegative responsiveness."

x over y also prefers z over w, and everybody who prefers w over z also prefers y over x, then if society prefers x to y, it should also prefer w to z.

(3.b) **Neutrality/monotonicity**. For all x, y, z, and w, assume that for all i, xP_iy implies wP_iz, and that for all i, zP_iw implies yP_ix. Then xPy implies wPz.

This strengthening of the neutrality assumption does not, by itself, give us an Arrow impossibility theorem when there are two or more people. In Example 9 below there are three people and four alternatives, a, b, c and d. The preferences of individuals 1, 2, and 3 are shown in the first 3 columns of the table. The fourth column shows social preferences under majority rule, which is used here, as in Examples 1 and 5, to generate the social preference relation.

Person 1	Person 2	Person 3	Society (Majority Rule)
a	c	a	a
b	a	c	c
c	b	d	b
d	d	b	d

Example 9. None. The complete and transitive social preferences assumption is satisfied, as are Pareto, neutrality/monotonicity, simple diversity, and no dictator. Majority rule works fine. There is no Arrow impossibility.

Example 9 shows that when $n \geq 2$ there is no Arrow impossibility, under the assumptions of complete and transitive social preferences, Pareto, neutrality/monotonicity, simple diversity, and no dictator.

5.7 Diversity

In this section we will modify the diverse preferences assumption.

Before doing so, let's revisit the assumption in the two-person world. In that world, simple diversity says there must exist a triple of alternatives x, y, z, such that xP_iy for $i = 1, 2$, but such that opinions are split on x versus z and on y versus z. That is, one person prefers x to z, while the other prefers z to x, and one person prefers y to z, while the other prefers z to y. Given our assumption that individual preferences are transitive, it must be the case that the two people's preferences over the triple can be represented as follows:

Person i	Person j
x	z
y	x
z	y

Table 5.1: Simple diversity array, $n = 2$.

Note that this is exactly the preference profile pattern of Examples 1, 2, 3 and 4. [6]

A somewhat similar array was used by Arrow in the proof of his impossibility theorem.[7] For now assume that V is any non-empty set of people in society, that V^C is the complement of V, and that V can be partitioned into two non-empty subsets V_1 and V_2. (Note that V^C may be empty.) The standard Arrow preference array looks like this:

Now, let's return to the question of how to modify the diverse preferences assumption. Example 9 shows that we cannot stick with the simple diversity array and still get an impossibility result. We

[6] Readers familiar with social choice theory will recognize the simple diversity array as being two thirds of the Condorcet voting paradox array. Condorcet's array simply adds a third person, say k, who prefers y to z to x.

[7] The array to which we now turn has been used by Arrow ([Ar2, p. 58]) and by many others since, including us ([FS, p. 294]).

People in V_1	People in V_2	People in V^C
x	z	y
y	x	z
z	y	x

Table 5.2: Standard Arrow array.

might start with the Condorcet voting paradox array, but if $n \geq 4$, we would have to worry about the preferences of people other than i, j and k. That suggests using something like the standard Arrow array. However, assuming the existence of a triple x, y, and z, and preferences as per that array, for *every subset of people* V *and every partition of* V, is an unnecessarily strong diversity assumption.

An even stronger diversity assumption was in fact used by Parks ([Pa]), Pollak and other originators of single-profile Arrow theorems. Pollak ([Po]) is clearest in his definition. His condition of "unrestricted domain over triples" requires the following: Imagine "any logically possible sub-profile" of individual preferences over three "hypothetical" alternatives x, y and z. Then there exist three actual alternatives a, b and c for which the sub-profile of preferences exactly matches that "logically possible sub-profile" over x, y and z. We will call this **Pollak diversity**. Let us consider what this assumption requires in the simple world of strict preferences, two people, and three alternatives. Pollak diversity would require that every one of the following arrays be represented, somewhere in the actual preference profile of the two people over the actual alternatives:

1	2	1	2	1	2	1	2	1	2	1	2
x	x	x	x	x	y	x	y	x	z	x	z
y	y	y	z	y	x	y	z	y	x	y	y
z	z	z	y	z	z	z	x	z	y	z	x

Table 5.3: Pollak diversity arrays, $n = 2$.

Note that the number of arrays in the table above is $3! = 6$. If n were equal to 3 we would have triples of columns instead of pairs, and there would have to be $(3!)^2 = 36$ such triples. With n people, the number of required n-tuples would be $(3!)^{n-1}$. In short, the number of arrays required for Pollak diversity rises exponentially with n. The number of alternatives rises with the number of required arrays, although not as fast because of array overlaps. Parks ([Pa]) uses an assumption ("diversity in society") that is very similar to Pollak's, although not so clear, and he indicates that it "requires at least 3^n alternatives…".

Pollak diversity is actually much stronger than necessary. We will weaken it as follows. We will not assume the existence of a triple x, y and z and every conceivable array of preferences on that triple. Nor will we assume the existence of a triple x, y and z and every conceivable array of preferences on that triple, but restricted to sets V, V_1, V_2, and V^C, as per the description of the standard Arrow array. Rather, we will simply assume the existence of triple x, y and z, and the standard Arrow array preferences on that triple, when it really matters. For our purposes, it really matters when the set V referenced in the description of the standard Arrow array is a decisive set. This is defined as follows:

Definition 10. We say that a set of people V is **decisive** if it is non-empty and if, for all alternatives x and y, if xP_iy for all i in V, then xPy.

It is appropriate to make a few comments about the notion of decisiveness. First, note that if person i is a dictator, then i by himself is a decisive set, and any set containing i is also decisive. Also, note that the Pareto principle (weak or strong) implies the set of all people is decisive. Second, in a multi-preference profile world, decisiveness for V would be a far stronger assumption that it is in the single-profile world, since it would require that (the same) V prevail no matter how preferences might change. We only require that V always prevail under the single preference profile.

Our diversity assumption is now modified as follows:

(5.b) **Complex diversity**. For any decisive set V with 2 or more members, there exists a triple of alternatives x, y, z, such that xP_iy for all i in V; such that yP_iz and zP_ix for everyone outside of V; and such that V can be partitioned into non-empty subsets V_1 and V_2, where the members of V_1 all put z last in their rankings over the triple, and the members of V_2 all put z first in their rankings over the triple.

The assumption of complex diversity means that for any decisive set V with two or more members, there is a triple x, y, and z, and a partition of V, which produces exactly the standard Arrow array shown above.

Simple diversity and complex diversity are related in the following way: If $n = 2$ and weak Pareto holds, they are equivalent. If $n > 2$, neither one implies the other, but they are both implied by Pollack diversity.

Referring back to Example 9 of the previous section, consider persons 2 and 3. Under simple majority rule, which was assumed in the example, they constitute a decisive coalition. However the complex diversity assumption fails in the example, because there is no way to define the triple x, y, z so as to get the standard Arrow array, when $V = \{2, 3\}$. Therefore complex diversity rules out that example.

Example 11 below modifies Example 9 so that, for the decisive set $V = \{2, 3\}$, the preference profile is consistent with complex diversity. (This example is created from Example 9 by switching alternatives a and b in person 3's ranking. Let $V_1 = \{2\}$, $V_2 = \{3\}$, and $V^C = \{1\}$. The triple x, y, z is now c, a, b.) Now that preferences have been modified consistent with our new diversity assumption, an Arrow impossibility pops up.

Person 1	Person 2	Person 3	Society (Majority Rule)
a	c	b	
b	a	c	aPb, bPc, cPa
c	b	d	aPd, bPd, cPd
d	d	a	

Example 11. Transitivity for social preference fails with a strict social preference cycle among a, b, and c. Society prefers a to b, b to c, and, irrationally, c to a.

Example 11 could be further modified by dropping alternative d, in which case it would become the Condorcet voting paradox array. (See footnote 6 above.) It would then have three people and three alternatives, and would satisfy complex diversity. Recall that Pollack diversity in the three-person case would require at least 36 n-tuples of alternatives, and that Parks diversity would require at least $3^n = 27$ alternatives. The point here is that that complex diversity is a much less demanding assumption, and requires many fewer alternatives, than Pollack diversity.

Complex diversity captures the idea of moderately divergent opinions when there are three or more people in society. It requires that when V is a decisive set with two or more members, there must exist some triple of alternatives x, y, and z about which there is basic disagreement, both within V (with those in V_1 putting z at the bottom and those in V_2 putting z and the top), and between V and V^C (with those in V preferring x over y, while those in V^C preferring y over x). But it is not an overly strong assumption, like Pollak diversity, nor does it require an enormous number of alternatives. We do not claim that the complex diversity assumption has the moral appeal of the Pareto principal or the no dictatorship assumption, but it is a plausible possibility, and one can very easily imagine real examples of preferences like those assumed in Example 11 above.

We will finish this discussion of diversity by noting our complex diversity assumption might be modified in either of two directions: It could be strengthened, by dropping the requirement in the definition that V be a decisive set. We will call the diversity assumption so modified **arbitrary *V* complex diversity**. This assumption would be closer to Pollak diversity. Alternatively, the complex diversity assumption could be weakened, by adding the requirement that V be a decisive set of minimal size. We will call the diversity assumption so modified **minimally-sized decisive V complex diversity**. We will briefly refer to both of these modifications at the end of the next section.

5.8 Arrow/Pollak Impossibility, $n \geq 2$

We now proceed to a proof of our second single-profile Arrow's theorem, which, unlike Theorem 8, is not restricted to a two-person society.[8] Although Pollak made a much stronger diversity assumption than we use, and although Parks ([Pa]), Hammond ([Ha]), and Kemp and Ng ([KN]), preceded Pollak with single-profile Arrow theorems, we will call this the Arrow/Pollak Impossibility Theorem, because of the similarity of our proof to his. But first, we need a proposition paralleling Proposition 7:

Proposition 12. *Assume $n \geq 2$ and neutrality/monotonicity. Assume there is a non-empty group of people V and a pair of alternatives x and y, such that xP_iy for all i in V and yP_ix for all i not in V. Suppose that xPy. Then V is decisive.*

Proof. This follows immediately from neutrality/monotonicity. □

Theorem 13 (Arrow/Pollak Impossibility Theorem)**.** *Assume $n \geq 2$. The assumptions of complete and transitive social preferences, weak Pareto, neutrality/monotonicity, complex diversity, and no dictator are mutually inconsistent.*

Proof. By the weak Pareto principle, the set of all individuals is decisive. Therefore decisive sets exist. Let V be a decisive set of minimal size, that is, a decisive set with no proper subsets that are also decisive. We will show that there is only one person in V, which will make that person a dictator. This will establish Arrow's theorem.

Suppose to the contrary that V has 2 or more members. By the complex diversity assumption there is a triple of alternatives x, y, and z, and a partition of V into non-empty subsets V_1 and V_2, giving the standard Arrow array as shown above. Since V is decisive, it must be true that xPy. Next we consider the social preference for x versus z.
Case 1. Suppose zRx. Then zPy by transitivity. Then V_2 becomes decisive by Proposition 12 above. But this is a contradiction, since we assumed that V was a decisive set of minimal size.
Case 2. Suppose not zRx. Then the social preference must be xPz, by completeness. But in this case V_1 is getting its way in the face of opposition by everone else, and by Proposition 12 above V_1 is decisive, another contradiction. □

In Section 5.7 above we mentioned two alternative versions of complex diversity, a stronger version, arbitrary V complex diversity, and a weaker version, minimally-sized decisive V complex diversity. Either of these could be substituted for complex diversity in Theorem 13 above, without affecting the proof. Moreover, using the minimally-sized V complex diversity assumption would give the following near-converse to Arrow's theorem: If there is a dictator, then the minimally-sized V complex diversity assumption is satisfied. This follows immediately from the definition of minimally-sized V complex diversity. For if i is a dictator, then $\{i\}$ is a decisive set; so any minimally-sized decisive set can have only one member, and therefore cannot be partitioned into two non-empty subsets. Consequently the definition of minimally-sized V complex diversity is vacuously satisfied.

5.9 Innocuous Dictators

In the standard multi-profile world, where all preference profiles are allowed (the so-called "universality," or "full domain" assumption) a dictator is a very bad thing indeed. A dictator in such a world forces his (strict) preference for x over y even if everyone else prefers y over x. In our single-profile world, on the other hand, a dictator may be innocuous. For instance, if person i is indifferent between all pairs of alternatives, he is by definition a dictator, although a completely benign one. Or, if everyone has exactly the same preferences over the alternatives, and weak Pareto

[8] There is a similar proof, but for a multi-profile Arrow's theorem, in Feldman & Serrano ([FS]).

is satisfied, then each person is a dictator. Or, if in a committee of five people, three have identical preferences, and if they use majority rule, then the three with identical preferences are all dictators. (Note however that in a standard median voter model, the median voter is not necessarily a dictator. While his favorite alternative may be the choice of the committee, the committee's preferences over all pairs of alternatives will not necessarily agree with his preferences over those pairs of alternatives.)

Therefore we need to make a final comment about why dictatorship should worry us, even though some dictators are innocuous: While we assume a single-profile world in this paper, and while for certain given profiles dictatorship doesn't look bad, we must remember that there can be other single-profile worlds with different given preference profiles. So, while in some cases an innocuous dictatorship is acceptable, in many other cases it is very much unacceptable. Moreover, both of our diversity assumptions exclude vacuous dictatorship cases like the one in which all individuals have exactly the same preferences. In sum, even though single-profile analysis may permit innocuous dictators, dictatorship remains a very bad thing, and Arrow's theorem remains important.

5.10 Conclusions

We have presented two new single-profile Arrow impossibility theorems which are simple and transparent. Theorem 8, which requires that there are only two people, relies on a very simple and modest assumption about diversity of preferences within the given preference profile, and on a relatively modest neutrality assumption. Theorem 13, which allows for two or more people, uses a substantially more complex assumption about diversity of preferences within the given profile, and uses a stronger neutrality/monotonicity assumption. Both theorems establish that Arrow impossibility happens even if individual preferences about alternatives are given and fixed.

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References

[Ar1] Kenneth Arrow: A Difficulty in the Concept of Social Welfare, *J. of Polit. Econ.* **58** (1950), 328–346.

[Ar2] Kenneth Arrow: *Social Choice and Individual Values*, 2nd ed. New York: John Wiley & Sons 1963.

[BD] Julian Blau and Rajat Deb: Social Decision Functions and the Veto, *Econometrica* **45** (1977), 871–879.

[Be] Abram Bergson: A Reformulation of Certain Aspects of Welfare Economics, *Quar. J. of Econ.* **52** (1938), 310–334.

[CK] Donald Campbell and Jerry Kelly: Impossibility Theorems in the Arrovian Framework, in Kenneth Arrow, Amartya Sen, and Kotaro Suzumura: *Handbook of Social Choice and Welfare*, vol. 1. Amsterdam: Elsevier Science 2002.

[DM] William Day and F. R. McMorris: Axiomatic Consensus Theory in Group Choice and Biomathematics, *SIAM* Philadelphia (2003)

[FS] Allan Feldman and Roberto Serrano: *Welfare Economics and Social Choice Theory*, 2nd ed. New York: Springer 2006.

[FM] Marc Fleurbaey and Philippe Mongin: The news of the death of welfare economics is greatly exaggerated, *Social Choice and Welfare* **25** (2005), 381–418.

[Ge] John Geanakoplos: Three Brief Proofs of Arrow's Impossibility Theorem, *Econ. Theory* **26** (2005), 211–215.

[Ha] Peter Hammond: Why Ethical Measures of Inequality Need Interpersonal Comparisons, *Theory and Decision* **7** (1976), 263–274.

[KN] Murray Kemp and Yew-Kwang Ng: On the Existence of Social Welfare Functions, Social Orderings and Social Decision Functions, *Economica* **43** (1976), 59–66.

[Pa] Robert Parks: An Impossibility Theorem for Fixed Preferences: A Dictatorial Bergson-Samuelson Welfare Function, *Review of Econ. Stud.* **43** (1976), 447–450.

[Po] Robert Pollak: Bergson-Samuelson Social Welfare Functions and the Theory of Social Choice, *Quar. J. of Econ.* **93** (1979), 73–90.

[Re] Philip Reny: Arrow's Theorem and the Gibbard-Satterthwaite Theorem: a Unified Approach, *Econ. Letters* **70** (2001), 99–105.

[Ro] Kevin Roberts: Social Choice Theory: The Single-profile and Multi-profile Approaches, *Review of Econ. Stud.* **47** (1980), 441–450.

[Ru] Ariel Rubinstein: The Single Profile Analogues to Multi-Profile Theorems: Mathematical Logic's Approach, *International Econ. Review* **25** (1984), 719–730.

[Sa1] Paul Samuelson: Arrow's Mathematical Politics, in S. Hook, ed.: *Human Values and Economics Policy*. New York: New York University Press 1967, 41–52.

[Sa2] Paul Samuelson: Reaffirming the Existence of 'Reasonable' Bergson-Samuelson Social Welfare Functions, *Economica* **44** (1977), 81–88.

[Se] Amartya Sen: Social Choice Theory: A Re-Examination, *Econometrica* **45** (1977), 53–89.

[Su] Kotaro Suzumura: Introduction, in Kenneth Arrow, Amartya Sen, and Kotaro Suzumura, eds.: *Handbook of Social Choice and Welfare*, vol. 1. Amsterdam: Elsevier Science 2002.

[Ub] Luis Ubeda: Neutrality in Arrow and Other Impossibility Theorems, *Econ. Theory* **23** (2004), 195–204.