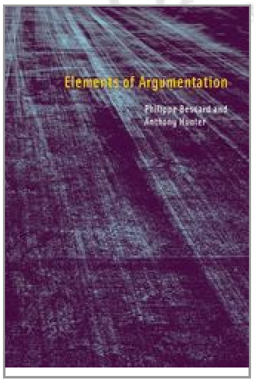

Elements of Argumentation

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Logical Argumentation

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[–] Abstract and Keywords

This chapter proposes a framework for modeling argumentation. The key features of this framework are the clarification of the nature of arguments and counterarguments; the identification of canonical undercuts, which are the only undercuts that need to be taken into account; and the representation of argument trees and argument structures which provide a way of exhaustively collating arguments and counterarguments.

Keywords: arguments, counterarguments, canonical undercuts, argument trees

To move beyond abstract argumentation, we introduce in this chapter a framework for argumentation in which more details about each argument are considered. In so doing, we distinguish the reasons (i.e., premises), the claim, and the method of inference by which the claim is meant to follow from the reasons. The nature of inference is diverse and includes analogical inference, causal inference, and inductive inference.

We focus on deductive inference and hence on deductive arguments, that is, the claim is a deductively valid consequence of the reasons (the support). We investigate the formalization of such arguments in the setting of classical logic. Thus, our starting position is that a deductive argument consists of a claim entailed by a collection of statements such that the claim as well as the statements are denoted by formulae of classical logic and entailment is identified with deduction in classical logic.

In our framework, an argument is simply a pair where the first item in the pair is a minimal consistent set of formulae that proves the second item. That is, we account for the support and the claim of an argument, though we do not indicate the method of inference, since it does not differ from one argument to another: We only consider deductive arguments; hence, the method of inference for each and every argument is always entailment according to classical logic.

Most proposals for modeling argumentation in logic are very limited in the way that they combine arguments for and against a particular claim. A simple form of argumentation is that a claim follows if and only if there is an argument for the claim and no argument against the claim. In our approach, we check how each argument is challenged by other arguments and by recursion for these counterarguments. Technically, an argument is undercut if and only if some of the reasons for the argument are rebutted (the reader may note that "undercut" is given a different meaning by some authors). Each undercut to a counterargument is itself an **(p.38)** argument and so may be undercut, and so by recursion each undercut needs to be considered. Exploring systematically the universe of arguments in order to present an exhaustive synthesis of the relevant chains of undercuts for a given argument is the basic principle of our approach.

Following this, our notion of an argument tree is that it is a synthesis of all the arguments that challenge the argument at the root of the tree, and it also contains all counterarguments that challenge these arguments, and so on, recursively.

3.1 Preliminaries

Prior to any definitions, we first assume a fixed Δ (a finite set of formulae) and use this Δ throughout. We further assume that every subset of Δ is given an enumeration $\langle\alpha_1, \ldots, \alpha_n\rangle$ of its elements, which we call its canonical enumeration. This really is not a demanding constraint: In particular, the constraint is satisfied whenever we impose an arbitrary total ordering over Δ. Importantly, the order has no meaning and is not meant to represent any respective importance of formulae in Δ. It is only a convenient way to indicate the order in which we assume the formulae in any subset of Δ are conjoined to make a formula logically equivalent to that subset.

The paradigm for our approach is a large repository of information, represented by Δ, from which arguments can be constructed for and against arbitrary claims. Apart from information being understood as declarative statements, there is no a priori restriction on the contents and the pieces of information in the repository can be arbitrarily complex. Therefore, Δ is not expected to be consistent. It need even not be the case that individual formulae in Δ are consistent.

The formulae in Δ can represent certain or uncertain information, and they can represent objective, subjective, or hypothetical statements as suggested in chapter 1. Thus, Δ can represent facts, beliefs, views, … Furthermore, the items in Δ can be beliefs from different agents who need not even have the same opinions. It can indeed be the case that an argument formed from such a Δ takes advantage of partial views from different agents. In any case, it is quite possible for Δ to have two or more formulae that are logically equivalent (e.g., Δ can be such that it contains both $\alpha \vee \beta$ and $\beta \vee \alpha$). However, wherever they come from, all formulae in Δ are on a par and treated equitably.

Note that we do not assume any metalevel information about formulae. In particular, we do not assume some preference ordering or "certainty ordering" over formulae. This is in contrast to numerous proposals **(p.39)** for argumentation that do assume some form of ordering over formulae. Such orderings can be useful to resolve conflicts by, for example, selecting formulae from a more reliable source. However, this, in a sense, pushes the problem of dealing with conflicting information to one of finding and using orderings over formulae, and as such raises further questions such as the following: Where does the knowledge about reliability of the sources come from? How can it be assessed? How can it be validated? Besides, reliability is not universal; it usually comes in specialized instances. This is not to say priorities are not useful. Indeed it is important to use them in some situations when they are available, but we believe that to understand the elements of argumentation, we need to avoid drawing on them—we need to have a comprehensive framework for argumentation that works without recourse to priorities over formulae.

3.2 Arguments

Here we adopt a very common intuitive notion of an argument and consider some of the ramifications of the definition. Essentially, an argument is a set of appropriate formulae that can be used to classically prove some claim, together with that claim (formulae represent statements, including claims).

Definition 3.2.1

An **argument** is a pair $\langle \Phi, \alpha \rangle$ such that

1. $\Phi \nvdash \bot$.
2. $\Phi \vdash \alpha$.
3. Φ is a minimal subset of Δ satisfying 2.

If $A = \langle \Phi, \alpha \rangle$ is an argument, we say that A is an argument for α (which in general is not an element of Δ) and we also say that Φ is a support for α. We call α the **claim** or the **consequent** of the argument, and we write Claim $(A) = \alpha$. We call Φ the **support** of the argument, and we write Support$(A) = \Phi$. Ω. denotes the set of all arguments, given Δ.

Example 3.2.1

Let $\Delta = \{\alpha, \alpha \rightarrow \beta, \gamma \rightarrow \neg\beta, \gamma, \delta, \delta \rightarrow \beta, \neg\alpha, \neg\gamma\}$. Some arguments are as follows:

$$\langle\{\alpha,\alpha\rightarrow\beta\},\beta\rangle$$
$$\langle\{\gamma\rightarrow\neg\beta,\gamma\},\neg\beta\rangle$$
$$\langle\{\delta,\delta\rightarrow\beta\},\beta\rangle$$
$$\langle\{\neg\alpha\},\neg\alpha\rangle$$
$$\langle\{\neg\gamma\},\neg\gamma\rangle$$

(p.40) The need for condition 1 of definition 3.2.1 can be illustrated by means of the next example, adapted from [Rea89].

Example 3.2.2

The story is that René Quiniou contends that John Slaney was in St-Malo on a certain day, and that Sophie Robin denies it. The situation is described by three statements, represented by the formulae to the left:

- $s \rightarrow q$ If Slaney was in St-Malo, Quiniou is right.
- $\neg(r \rightarrow q)$ It is not the case that if Robin is right, so is Quiniou.
- $\neg(s \rightarrow r)$ It is not the case that if Slaney was in St-Malo, Robin is right.

Intuitively, nothing there provides grounds for an argument claiming that I am the Pope (the latter statement is denoted p). Still, note that $\{\neg(r \rightarrow q), \neg(s \rightarrow r)\}$ is a minimal set of formulae satisfying condition 2 with respect to deducing p:

$$\langle\{\neg(r\rightarrow q),\neg(s\rightarrow r)\}\vdash p$$

If it were not for condition 1 that is violated because $\{\neg(r \rightarrow q), \neg(s \rightarrow r)\}$ is inconsistent,

$$\langle\{\neg(r\rightarrow q),\neg(s\rightarrow r)\},p\rangle$$

would be an argument in the sense of definition 3.2.1, to the effect that I am the Pope!

Condition 2 of definition 3.2.1 aims at ensuring that the support is sufficient for the consequent to hold, as is illustrated in the next example.

Example 3.2.3

The following is a sound deductive argument in free text.

Chemnitz can't host the Summer Olympics because it's a small city and it can host the Summer Olympics only if it is a major city.

Below is an attempt at formalizing the example:

- $o \to m$ Chemnitz can host the Summer Olympics only if Chemnitz is a major city.
- s Chemnitz is a small city.

Hence
- $\neg o$ Chemnitz cannot host the Summer Olympics.

(p.41) According to classical logic, the purported conclusion fails to follow from the premises

$$\{o \to m, s\} \nvdash \neg 0$$

because of a missing, implicit, statement:
- $s \to \neg m$ If Chemnitz is a small city, then Chemnitz is not a major city.

An enthymeme is a form of reasoning in which some premises are implicit, most often because they are obvious. For example, "The baby no longer has her parents; therefore, she is an orphan" (in symbols, $\neg p$ hence o) is an enthymeme: The reasoning is correct despite omitting the trivial premise stating that "if a baby no longer has her parents, then she is an orphan" (in symbols, $\neg p, \neg p \to o \vdash o$).

Example 3.2.3 shows that, by virtue of condition 2 in definition 3.2.1, arguments in the form of enthymemes are formalized with all components made explicit.

Remember that the method of inference from support to consequent is deduction according to classical logic, which explains the first two conditions in definition 3.2.1.

Minimality (i.e., condition 3 in definition 3.2.1) is not an absolute requirement, although some properties below depend on it. Importantly, the condition is not of a mere technical nature.

Example 3.2.4

Here are a few facts about me …

- p I like paprika.
- r I am retiring.
- $r \to q$ If I am retiring, then I must quit my job.

It is possible to argue "I must quit my job because I am retiring and if doing so, I must quit" to be captured formally by the argument

$$\langle\{r, r \to q\}, q\rangle$$

In contrast, it is counterintuitive to argue "I must quit my job because I like paprika and I am retiring and if doing so, I must quit" to be captured formally by

$$\langle\{p, r, r \to q\}, q\rangle$$

which fails to be an argument because condition 3 is not satisfied.

The underlying idea for condition 3 is that an argument makes explicit the connection between reasons for a claim and the claim itself. However, **(p.42)** that would not be the case if the reasons were not exactly identified—in other words, if reasons incorporated irrelevant information and so included formulae not used in the proof of the claim.

Arguments are not necessarily independent. In a sense, some encompass others (possibly up to some form of equivalence), which is the topic we now turn to.

Definition 3.2.2

An argument $\langle\Phi, \alpha\rangle$ is **more conservative** than an argument $\langle\Psi, \beta\rangle$ iff $\Phi \subseteq \Psi$ and $\beta \vdash \alpha$.

Example 3.2.5

$\langle\{\alpha\}, \alpha \vee \beta\rangle$ is more conservative than $\langle\{\alpha, \alpha \to \beta\}, \beta\rangle$. Here, the latter argument can be obtained from the former (using $\alpha \to \beta$ as an extra hypothesis), but the reader is warned that this is not the case in general (see proposition 3.2.4).

Roughly speaking, a more conservative argument is more general: It is, so to speak, less demanding on the support and less specific about the consequent.

Example 3.2.6

Now we consider some ruminations on wealth:

- p I own a private jet.
- $p \to q$ If I own a private jet, then I can afford a house in ueens Gardens.

Hence "I can afford a house in Queens Gardens," which is captured by the following argument:

$$\langle\{p, p \to q\}, q\rangle$$

Let the same two considerations be supplemented with another one:

- p I own a private jet.
- $p \rightarrow q$ If I own a private jet, then I can afford a house in Queens Gardens.
- $q \rightarrow r$ If I can afford a house in Queens Gardens, then I am rich.

Hence "I am rich, and I can afford a house in Queens Gardens," which is captured by the following argument:

$$\langle\{p, p \rightarrow q, q \rightarrow r\}, r \wedge q\rangle$$

However, the previous argument $\langle\{p, p \rightarrow q\}, q\rangle$ is more conservative than $\langle\{p, p \rightarrow q, q \rightarrow r\}, r \wedge q\rangle$, which can somehow be retrieved from it:

$$\left.\begin{array}{l}\langle\{p, p \rightarrow q\}, q\rangle \\ \{q, q \rightarrow r\} \models r \wedge q\end{array}\right\} \Rightarrow \langle\{p, p \rightarrow q, q \rightarrow r\}, r \wedge q\rangle$$

(p.43) The next section is devoted to a number of formal properties and can be skipped if desired.

3.2.1 Technical Developments

The subsections entitled "Technical developments" provide more detailed consideration of the framework and may be skipped on a first reading. All proofs are to be found in appendix D.

Based on refutation, there is a simple characterization of being an argument.

Proposition 3.2.1

$\langle\Phi, \alpha\rangle$ is an argument iff $\Phi \cup \{\neg\alpha\}$ is a minimal inconsistent set such that $\Phi \subseteq \Delta$.

What requirements are needed to make a new argument out of an existing one? The first possibility deals with the case that the new argument only differs by its consequent.

Proposition 3.2.2

Let $\langle\Phi, \alpha\rangle$ be an argument. If $\Phi \vdash \alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$ a is a tautology, then $\langle\Phi, \beta\rangle$ is also an argument.

It is not possible to loosen the conditions in proposition 3.2.2. Taking $\Phi = \{\beta, \alpha \leftrightarrow \beta\}$ gives an argument $\langle\Phi, \alpha\rangle$ such that $\Phi \vdash \alpha \rightarrow \beta$ and $\Phi \vdash \beta \rightarrow \alpha$, but $\langle\Phi, \beta\rangle$ fails to be an argument.

Proposition 3.2.3

Let Φ and Ψ be such that there exists a bijection f from Ψ to some partition $\{\Phi_1, \ldots \Phi_n\}$ of Φ where $\mathrm{Cn}(\{\psi\}) = \mathrm{Cn}(f(\psi))$ for all $\psi \in \Psi$. If $\langle\Phi, \alpha\rangle$ is an argument, then $\langle\Psi, \alpha\rangle$ is also an argument.

The converse of proposition 3.2.3 fails: Let $\Delta = \{\alpha, \alpha \to \beta, \alpha \wedge \beta\}$. Let $\Psi = \{\alpha \wedge \beta\}$ and $\Phi = \{\alpha, \alpha \to \beta\}$. Now, $\langle\Psi, \alpha\rangle$ is an argument, but $\langle\Phi, \alpha\rangle$ is not.

Corollary 3.2.1

Let $\Phi = \{\phi_1, \ldots, \phi_n\} \subseteq \Delta$ and $\Psi = \{\psi_1, \ldots, \psi_n\} \subseteq \Delta$ such that $\varphi_i \leftrightarrow \psi_i$ is a tautology for $i = 1 \ldots n$. Let α and β be such that $\alpha \leftrightarrow \beta$ is a tautology. Then, $\langle\Phi, \alpha\rangle$ is an argument iff $\langle\Psi, \beta\rangle$ is an argument.

It is not possible to extend corollary 3.2.1 substantially. Clearly, proposition 3.2.3 can neither be extended to the case $\mathrm{Cn}(\{\psi\}) \subseteq \mathrm{Cn}(f(\psi))$ (if $\Phi = \{\alpha\}$ and $\Psi = \{\alpha, \vee\, \beta\}$ then $\langle\Phi, \alpha\rangle$ is an argument, but $\langle\Phi, \alpha\rangle$ is not) nor to the case $\mathrm{Cn}(f(\psi)) \subseteq \mathrm{Cn}(\{\psi\})$ (if $\Phi = \{\beta \wedge \gamma, \beta \wedge \gamma \to \alpha\}$ and $\Psi = \{\alpha \wedge \beta \wedge \gamma, \alpha \wedge \delta\}$ then $\langle\Phi, \alpha\rangle$ is an argument, but $\langle\Psi, \beta\rangle$ is not).

Example 3.2.5 suggests that an argument $\langle\Psi, \beta\rangle$ can be obtained from a more conservative argument $\langle\Phi, \alpha\rangle$ by using $\Psi\backslash\Phi$ together with α in order **(p.44)** to deduce β (in symbols, $\{\alpha\} \cup \Psi\backslash\Phi \vdash \beta$ or, equivalently, $\Psi\backslash\Phi \vdash \alpha \to \beta$). As already mentioned, this does not hold in full generality. A counterexample consists of $\langle\{\alpha \wedge \gamma\}, \alpha\rangle$ and $\langle\{\alpha \wedge \gamma, \neg\alpha \vee \beta \vee \neg\gamma\}, \beta\rangle$. However, a weaker property holds.

Proposition 3.2.4

If $\langle\Phi, \alpha\rangle$ is more conservative than $\langle\Psi, \beta\rangle$, then $\Psi\backslash\Phi \vdash \varphi \to (\alpha \to \beta)$ for some formula φ such that $\Phi \vdash \varphi$ and $\varphi \nvdash \alpha$ unless α is a tautology.

The interesting case, as in example 3.2.5, is when φ can be a tautology.

What is the kind of structure formed by the set of all arguments? That some arguments are more conservative than others provides the basis for an interesting answer.

Proposition 3.2.5

Being more conservative defines a pre-ordering over arguments. Minimal arguments always exist unless all formulae in Δ are inconsistent. Maximal arguments always exist: They are $\left\langle \emptyset, \top \right\rangle$, where $\top$ is any tautology.

A useful notion is then that of a normal form (a function such that any formula is mapped to a logically equivalent formula and, if understood in a strict sense as here, such that any two logically equivalent formulae are mapped to the same formula).

Proposition 3.2.6

Given a normal form, being more conservative defines an ordering provided that only arguments that have a consequent in normal form are considered. The ordered set of all such arguments is an upper semilattice (when restricted to the language of Δ). The greatest argument always exists; it is $\langle \emptyset, \top \rangle$.

Example 3.2.7

The greatest lower bound of $\langle \{\alpha \wedge \beta\}, \alpha \rangle$ and $\langle \{\alpha \wedge \neg\beta\}, \alpha \rangle$ does not exist. If $\Delta = \{\alpha \wedge \beta, \alpha \wedge \neg\beta\}$ then there is no least argument. Taking now $\Delta = \{\alpha, \beta, \alpha \leftrightarrow \beta\}$ there is no least argument either (although Δ is consistent). Even though $\Delta = \{\alpha, \beta \wedge \neg\beta\}$ is inconsistent, the least argument exists: $\langle \{\alpha\}, \alpha' \rangle$ (where α' stands for the normal form of α). As the last illustration, $\Delta = \{\alpha \vee \beta, \beta\}$ admits the least argument $\langle \{\beta\}, \beta' \rangle$ (where β' stands for the normal form of β).

In any case, $\langle \emptyset, \top \rangle$ is more conservative than any other argument.

The notion of being more conservative induces a notion of equivalence between arguments. However, another basis for equating two arguments with each other comes to mind: pairwise logical equivalence of the components of both arguments.

(p.45) Definition 3.2.3

Two arguments $\langle \Phi, \alpha \rangle$ and $\langle \Psi, \beta \rangle$ are **equivalent** iff Φ is logically equivalent to Ψ and α is logically equivalent to β.

Proposition 3.2.7

Two arguments are equivalent whenever each is more conservative than the other. In partial converse, if two arguments are equivalent, then either each is more conservative than the other or neither is.

Thus, there exist equivalent arguments $\langle \Phi, \alpha \rangle$ and $\langle \Psi, \beta \rangle$ that fail to be more conservative than each other (as in example 3.2.8 below). However, if $\langle \Phi, \alpha \rangle$ is strictly more conservative than $\langle \Psi, \beta \rangle$ (meaning that $\langle \Phi, \alpha \rangle$ is more conservative than $\langle \Psi, \beta \rangle$, but $\langle \Psi, \beta \rangle$ is not more conservative than $\langle \Phi, \alpha \rangle$), then $\langle \Phi, \alpha \rangle$ and $\langle \Psi, \beta \rangle$ are not equivalent.

Example 3.2.8

Let $\Phi = \{\alpha, \beta\}$ and $\Psi = \{\alpha \vee \beta, \alpha \leftrightarrow \beta\}$. The arguments $\langle \Phi, \alpha \wedge \beta \rangle$ and $\langle \Psi, \alpha \wedge \beta \rangle$ are equivalent even though neither is more conservative than the other. This means that there exist two distinct subsets of Δ (namely, Φ and Ψ) supporting a $\alpha \wedge \beta$.

While equivalent arguments make the same point (i.e., the same inference), we do want to distinguish equivalent arguments from each other. What we do not want is to distinguish between arguments that are more conservative than each other.

3.3 Defeaters, Rebuttals, and Undercuts

An intuitive idea of counterargument is captured with the notion of defeaters, which are arguments whose claim refutes the support of another argument [FKEG93, Nut94, Vre97, Ver99]. This gives us a general way for an argument to challenge another.

Definition 3.3.1

A **defeater** for an argument $\langle \Phi, \alpha \rangle$ is an argument $\langle \Psi, \beta \rangle$ such that $\beta \vdash \neg (\phi_1 \wedge \ldots \wedge \phi_n)$ for some $\{\phi_1, \ldots \phi_n\} \subseteq \Phi$.

Example 3.3.1

Let $\Delta = \{\neg\alpha, \alpha \vee \beta, \alpha \leftrightarrow \beta, \gamma \rightarrow \alpha\}$. Then, $\langle \{\alpha \vee \beta, \alpha \leftrightarrow \beta\}, \alpha \wedge \beta \rangle$ is a defeater for $\langle \{\neg\alpha, \gamma \rightarrow \alpha\}, \neg\gamma \rangle$. A more conservative defeater for $\langle \{\neg\alpha, \gamma \rightarrow \alpha\}$ is $\langle \{\alpha \vee \beta, \alpha \leftrightarrow \beta\}, \alpha \vee \gamma \rangle$.

The notion of assumption attack to be found in the literature is less general than the above notion of defeater, of which special cases are undercut and rebuttal as discussed next.

Some arguments directly oppose the support of others, which amounts to the notion of an undercut.

(p.46) Definition 3.3.2

An **undercut** for an argument $\langle \Phi, \alpha \rangle$ is an argument $\langle \Psi, \neg(\phi_1 \wedge \ldots \wedge \phi_n) \rangle$ where $\{\phi_1, \ldots, \phi_n\} \subseteq \Phi$.

Example 3.3.2

Let $\Delta = \{\alpha, \alpha \rightarrow \beta, \gamma, \gamma \rightarrow \neg\alpha\}$. Then, $\langle \{\gamma, \gamma \rightarrow \neg \alpha\}, \neg\{\alpha \wedge (\alpha \rightarrow \beta)\} \rangle$ is an undercut for $\langle \{\alpha, \alpha \rightarrow \beta\}, \beta \rangle$. A less conservative undercut for $\langle \{\alpha, \alpha \rightarrow \beta\}, \beta \rangle$ is $\langle \{\gamma, \gamma \rightarrow \neg\alpha\}, \neg\alpha \rangle$.

Presumably, the most direct form of a conflict between arguments is when two arguments have opposite claims. This case is captured in the literature through the notion of a rebuttal.

Definition 3.3.3

An argument $\langle \Psi, \beta \rangle$ is a **rebuttal** for an argument $\langle \Phi, \alpha \rangle$, ai iff $\beta \leftrightarrow \neg\alpha$ is a tautology.

Example 3.3.3

We now return to the Simon Jones affair in five statements:

- p Simon Jones is a Member of Parliament.

- $p \to \neg q$ If Simon Jones is a Member of Parliament, then we need not keep quiet about details of his private life.
- r Simon Jones just resigned from the House of Commons.
- $r \to \neg p$ If Simon Jones just resigned from the House of Commons, then he is not a Member of Parliament.
- $\neg p \to q$ If Simon Jones is not a Member of Parliament, then we need to keep quiet about details of his private life.

The first two statements form an argument A whose claim is that we can publicize details about his private life. The next two statements form an argument whose claim is that he is not a Member of Parliament (contradicting an item in the support of A) and that is a counterargument against A. The last three statements combine to give an argument whose claim is that we cannot publicize details about his private life (contradicting the claim of A), and that, too, is a counterargument against A. In symbols, we obtain the following argument (below left) and counterarguments (below right).

$$\langle\{p, p \to \neg q\}, \neg q\rangle \quad \begin{cases} \text{An undercut is } \langle\{r, r \to \neg p\}, \neg p\rangle \\ \text{A rebuttal is } \langle\{r, r \to \neg p, \neg p \to q\}, q\rangle \end{cases}$$

Trivially, undercuts are defeaters, but it is also quite simple to establish the next result.

Proposition 3.3.1

If $\langle\Psi, \beta\rangle$ is a rebuttal for an argument $\langle\Phi, \alpha\rangle$, then $\langle\Psi, \beta\rangle$ is a defeater for $\langle\Phi, \alpha\rangle$.

(p.47) If an argument has defeaters, then it has undercuts, naturally. It may happen that an argument has defeaters but no rebuttals as illustrated next.

Example 3.3.4

Let $\Delta = \{\alpha \wedge \beta, \neg\beta\}$. Then, $\langle\{\alpha \wedge \beta\}, \alpha\rangle$ has at least one defeater but no rebuttal.

Here are some details on the differences between rebuttals and undercuts.

An undercut for an argument need not be a rebuttal for that argument

As a first illustration, $\langle\{\neg\alpha\}, \neg\alpha\rangle$ is an undercut for $\langle\{\alpha, \alpha \to \beta\}, \beta\rangle$ but is not a rebuttal for it. Clearly, $\langle\{\neg\alpha\}, \neg\alpha\rangle$ does not rule out β. Actually, an undercut may even agree with the consequent of the objected argument: $\langle\{\beta \wedge \neg\alpha\rangle$ is an undercut for $\langle\{\alpha, \alpha \to \beta\}, \beta\rangle$. In this case, we have an argument with an undercut that conflicts with the support of the argument but implicitly provides an alternative way to deduce the consequence of the argument (see the so-called overzealous arguments in section 5.3.1). This should make it clear that an undercut need not question the consequent of an argument but only the reason(s) given by that argument to support its consequent. Of course, there are also undercuts that challenge an argument on both counts: Just consider $\langle\{\neg\alpha \wedge \neg\beta\}, \neg\alpha\rangle$, which is such an undercut for $\langle\{\alpha, \alpha \to \beta\}, \beta\rangle$.

A rebuttal for an argument need not be an undercut for that argument

As an example, $\langle\{\neg\beta\}, \neg\beta\rangle$ is a rebuttal for $\langle\{\alpha, \alpha \to \beta\}, \beta\rangle$ but is not an undercut for it because β is not in $\{\alpha, \alpha \to \beta\}$. Observe that there is not even an argument equivalent to $\langle\{\neg\beta\}, \neg\beta\rangle$ that would be an undercut for $\langle\{\alpha, \alpha \to \beta\}, \beta\rangle$: In order to be an undercut for $\langle\{\alpha, \alpha \to \beta\}, \beta\rangle$, an argument should be of the form $\langle\Phi, \neg\alpha\rangle$, $\langle\Phi, \neg(\alpha \to \beta)\rangle$ or $\langle\Phi, \neg(\alpha \wedge (\alpha \to \beta))\rangle$, but $\neg\beta$ is not logically equivalent to $\neg\alpha$, $\neg(\alpha \to \beta)$ or $\neg(\alpha \wedge (\alpha \to \beta))$.

Anyway, a rebuttal for an argument is a less conservative version of a specific undercut for that argument as we now prove.

Proposition 3.3.2

If $\langle \Psi, \beta\rangle$ is a defeater for $\langle \Phi, \alpha \rangle$, then there exists an undercut for $\langle\Phi, \alpha\rangle$ that is more conservative than $\langle \Psi, \beta \rangle$.

Corollary 3.3.1

If $\langle\Psi, \beta\rangle$ is a rebuttal for $\langle\Psi, \alpha\rangle$, then there exists an undercut for $\langle\Phi, \alpha\rangle$ that is more conservative than $\langle\Psi, \beta\rangle$.

The undercut mentioned in proposition 3.3.2 and corollary 3.3.1 is strictly more conservative than $\langle\Psi, \beta$ whenever $\neg\beta$ fails to be logically equivalent with Φ. **(p.48)**

Figure 3.1 If A has a defeater, there is an infinite series of counterarguments, starting with a counterargument for A, and then for each counterargument, by recursion, there is a counterargument.

Proposition 3.3.3

If an argument has a defeater, then there exists an undercut for its defeater.

Using $\mathbb{N}$ to denote the nonnegative integers, a noticeable consequence can be stated as follows.

Corollary 3.3.2

If an argument A has at least one defeater, then there exists an infinite sequence of arguments $(A_n)_{n \in \mathbb{N}^*}$, such that A_1 is A and A_{n+1} is an undercut of A_n for every $n \in \mathbb{N}^*$.

As an illustration of corollary 3.3.2, see figure 3.1.

Example 3.3.5

Let $\Delta = \{\alpha, \alpha \rightarrow \beta, \gamma \wedge \neg\alpha\}$. Then,

$$\langle\{\alpha, \alpha \rightarrow \beta\}, \alpha \wedge \beta\rangle$$

is an argument for which a defeater is

$$\langle\{\gamma \wedge \neg\alpha\}, \gamma \wedge \neg\alpha\rangle$$

The argument $\langle\{\alpha, \alpha \rightarrow \beta\}, \alpha \wedge \beta\rangle$ gives rise to an infinite series of arguments where each one is an undercut for the previous argument in the series:

$$\begin{array}{c} \langle\{\alpha, \alpha \rightarrow \beta\}, \alpha \wedge \beta\rangle \\ \uparrow \\ \langle\{\gamma \wedge \neg\alpha\}, \neg\alpha\rangle \\ \uparrow \\ \langle\{\alpha\}, \neg(\gamma \wedge \neg\alpha)\rangle \\ \uparrow \\ \langle\{\gamma \wedge \neg\alpha\}, \neg\alpha\rangle \\ \vdots \end{array}$$

(p.49) Corollary 3.3.2 is obviously a potential concern for representing and comparing arguments. We address this question in section 3.5.

Section 2.1 introduced the notions of self-attacking arguments and controversial arguments. The next two results deal with such special arguments.

Proposition 3.3.4

Let $\langle\Phi, \alpha\rangle$ be an argument for which $\langle\Psi, \beta\rangle$ is a defeater. Then, $\Psi \not\subseteq \Phi$.

Proposition 3.3.4 proves that, in the sense of definition 3.2.1 and definition 3.3.1, no argument is self-defeating.

Proposition 3.3.5

If $\langle\Gamma, \neg\psi\rangle$ is an undercut for $\langle\Psi, \neg\phi\rangle$, which is itself an undercut for an argument $\langle\Phi, \alpha\rangle$, then $\langle\Gamma, \neg\psi\rangle$ is not a defeater for $\langle\Phi, \alpha\rangle$.

Proposition 3.3.5 shows that no undercut is controversial, again in the sense of definition 3.2.1 and definition 3.3.1. This does not extend to defeaters as illustrated by the following example.

Example 3.3.6

Consider the following arguments:

$$\begin{array}{ll} A_1 & \langle\{\alpha \wedge \neg\beta\}, \alpha\rangle \\ A_2 & \langle\{\neg\alpha \wedge \beta\}, \neg(\alpha \wedge \neg\beta)\rangle \\ A_3 & \langle\{\neg\alpha \wedge \neg\beta\}, \neg\alpha \wedge \neg\beta\rangle \end{array}$$

Thus, A_2 is an undercut for A_1 and A_3 is a defeater for A_2. Furthermore, A_3 is also a defeater for A_1.

As arguments can be ordered from more conservative to less conservative, there is a clear and unambiguous notion of maximally conservative defeaters for a given argument (the ones that are representative of all defeaters for that argument).

Definition 3.3.4

$\langle\Psi, \beta\rangle$ is a **maximally conservative defeater** of $\langle\Phi, \alpha\rangle$ iff for all defeaters $\langle\Psi', \beta'\rangle$ of $\langle\Phi, \alpha,\rangle$ if $\Psi' \subseteq \Psi$ and $\beta \vdash \beta'$, then $\Psi \subseteq \Psi'$ and $\beta' \subset \beta$.

Equivalently, $\langle\Psi, \beta\rangle$ is a maximally conservative defeater of $\langle\Phi, \alpha\rangle$ iff $\langle\Psi, \beta\rangle$ is a defeater of $\langle\Phi, \alpha\rangle$ such that no defeaters of $\langle\Phi, \alpha\rangle$ are strictly more conservative than $\langle\Psi, \beta\rangle$.

Proposition 3.3.6

If $\langle\Psi, \beta\rangle$ is a maximally conservative defeater of $\langle\Phi, \alpha\rangle$, then $\langle\Psi, \beta'\rangle$ is an undercut of $\langle\Phi, \alpha\rangle$ for some β' that is logically equivalent with β.

(p.50) Proposition 3.3.6 suggests that we focus on undercuts when seeking counterarguments to a given argument as is investigated from now on.

3.3.1 Technical Developments

The first question to be investigated here is under what condition can an argument defeat its defeaters?

Proposition 3.3.7

Given two arguments $\langle\Phi, \alpha\rangle$ and $\langle\Psi, \beta\rangle$ such that $\{\alpha, \beta\} \vdash \varphi$ for each $\varphi \in \Phi$ if $\langle\Psi, \beta\rangle$ is a defeater for $\langle\Phi, \alpha\rangle$, then $\langle\Phi, \alpha\rangle$ is a defeater for $\langle\Psi, \beta\rangle$.

Corollary 3.3.3

Let α be logically equivalent with Φ. If $\langle \Psi, \beta \rangle$ is a defeater for $\langle \Phi, \alpha \rangle$, then $\langle \Phi, \alpha \rangle$ is a defeater for $\langle \Psi, \beta \rangle$.

A follow-up is the case about rebuttals.

Corollary 3.3.4

If $\langle \Psi, \beta \rangle$ is a rebuttal for $\langle \Phi, \alpha \rangle$, then $\langle \Phi, \alpha \rangle$ is a rebuttal for $\langle \Psi, \beta \rangle$.

A similar question is when are two arguments a defeater of each other?

Proposition 3.3.8

Given two arguments $\langle \Phi, \alpha \rangle$ and $\langle \Psi, \beta \rangle$ such that $\neg\{\alpha \wedge \beta)$ is a tautology, $\langle \Psi, \beta \rangle$ is a defeater for $\langle \Phi, \alpha \rangle$, and $\langle \Phi, \alpha \rangle$ is a defeater for $\langle \Psi, \beta \rangle$.

While proposition 3.3.4 expresses that the defeat relation is anti-reflexive, proposition 3.3.7 and proposition 3.3.8 show that the defeat relation is symmetric on parts of the domain (i.e., it is symmetric for some arguments).

Returning to features of individual counterarguments, what does it take for a defeater to be a rebuttal?

Proposition 3.3.9

Let $\langle \Psi, \beta \rangle$ be a defeater for an argument $\langle \Phi, \alpha \rangle$. If $\alpha \vee \beta$ is a tautology and $\{\alpha, \beta\} \vdash \varphi$ for each $\varphi \in \Phi$, then $\langle \Psi, \beta \rangle$ is a rebuttal for $\rangle \Phi, \alpha \rangle$.

Proposition 3.3.10

If $\langle \Phi, \alpha \rangle$ is an argument where Φ is logically equivalent with α, then each defeater $\langle \Psi, \beta \rangle$ of $\langle \Phi, \alpha \rangle$ such that $\alpha \vee \beta$ is a tautology is a rebuttal for $\Phi, \alpha \rangle$.

Note: In proposition 3.3.10, the assumption that $\alpha \vee \beta$ is a tautology can be omitted when considering the alternative definition of a rebuttal where $\langle \Psi, \beta \rangle$ is a rebuttal for $\langle \Phi, \alpha \rangle$ iff $\neg\alpha \vee \neg\beta$ is a tautology (and the proof gets simpler, of course).

(p.51) It has been exemplified above that the notion of a rebuttal and the notion of an undercut are independent. The next result characterizes the cases in which both notions coincide.

Proposition 3.3.11

Let $\langle \Phi, \alpha \rangle$ and $\langle \Psi, \beta \rangle$ be two arguments. $\langle \Psi, \beta \rangle$ is both a rebuttal and an undercut for $\langle \Phi, \alpha \rangle$ iff Φ is logically equivalent with α and β is $\neg(\phi_1 \wedge \ldots \wedge \phi_n)$ such that $\Phi = \{\varphi_1 \ldots, \varphi_n\}$.

Interestingly enough, the support of a maximally conservative de-feater leaves no choice about its consequent as is proven in proposition 3.3.12.

Proposition 3.3.12

Let $\langle \Psi, \beta \rangle$ be a maximally conservative defeater for an argument $\langle \Phi, \alpha \rangle$. Then, $\langle \Psi, \gamma \rangle$ is a maximally conservative defeater for $\langle \Phi, \alpha \rangle$ iff γ is logically equivalent with β.

Proposition 3.3.12 does not extend to undercuts because they are syntax dependent (in an undercut, the consequent is always a formula governed by negation).

Lastly, in what way does the existence of a defeater relate to inconsistency for Δ?

Proposition 3.3.13

Δ is inconsistent if there exists an argument that has at least one defeater. Should there be some inconsistent formula in Δ, the converse is untrue. When no formula in Δ is inconsistent, the converse is true in the form: If Δ is inconsistent, then there exists an argument that has at least one rebuttal.

Corollary 3.3.5

Δ is inconsistent if there exists an argument that has at least one undercut. The converse is true when each formula in Δ is consistent.

3.4 Canonical Undercuts

As defined above, an undercut for an argument $\langle \Phi, \alpha \rangle$ is an argument $\langle \Psi, \neg(\phi_1 \wedge \ldots \wedge \phi_n) \rangle$ where $\{\phi_1, \ldots, \phi_n\} \subseteq \Phi$ and $\Phi \cup \Psi \subseteq \Delta$ by the definition of an argument.

While proposition 3.3.2 and proposition 3.3.6 point to undercuts as candidates to be representative of all defeaters for an argument, maximally conservative undercuts are even better candidates.

Definition 3.4.1

$\langle \Psi, \beta \rangle$ is a **maximally conservative undercut** of $\langle \Phi, \alpha \rangle$ iff for all undercuts $\langle \Psi', \beta' \rangle$ of $\langle \Phi, \alpha \rangle$, if $\Psi' \subseteq$ and $\beta \vdash \beta'$ then $\Psi \subseteq \Psi'$ and $\beta' \vdash \beta$.

(p.52) Evidently, $\langle \Psi, \beta \rangle$ is a maximally conservative undercut of $\langle \Phi, \alpha \rangle$ iff $\langle \Psi, \beta \rangle$ is an undercut of $\langle \Phi, \alpha \rangle$ such that no undercuts of $\langle \Phi, \alpha \rangle$ are strictly more conservative than $\langle \Psi, \beta \rangle$.

Stressing the relevance of maximally conservative undercuts, it can be proved (proposition 3.4.6) that each maximally conservative undercut is a maximally conservative defeater (but not vice versa, of course).

Example 3.4.1 now shows that a collection of counterarguments to the same argument can sometimes be summarized in the form of a single maximally conservative undercut of the argument, thereby avoiding some amount of redundancy among counterarguments.

Example 3.4.1

Consider the following formulae concerning who is going to a party:

- $r \rightarrow \neg p \wedge \neg q$ If Rachel goes, neither Paul nor Quincy goes.
- p Paul goes.
- q Quincy goes.

Hence both Paul and Quincy go (initial argument):

$$\langle\{p, q\}, p \wedge q\rangle$$

Now assume the following additional piece of information:

- r Rachel goes.

Hence Paul does not go (a first counterargument):

$$\langle\{r, r \rightarrow \neg p \wedge \neg q\}, \neg p\rangle$$

Hence Quincy does not go (a second counterargument):

$$\langle r, r \rightarrow \neg p \wedge \neg q\}, \neg q\rangle$$

A maximally conservative undercut (for the initial argument) that subsumes both counterarguments above is

$$\langle\{r, r \rightarrow \neg p \wedge \neg q\}, \neg(p \wedge q)\rangle$$

The fact that the maximally conservative undercut in example 3.4.1 happens to be a rebuttal of the argument is only accidental. Actually, the consequent of a maximally conservative undercut for an argument is exactly the negation of the full support of the argument.

Proposition 3.4.1

If $\langle\Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n\rangle$ is a maximally conservative undercut for an argument $\langle\Phi, \alpha\rangle$, then $\Phi = \{\phi, \ldots, \phi_n$.

(p.53) Note that if $\langle\Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n)\rangle$ is a maximally conservative undercut for an argument $\langle\Phi, \alpha\rangle$, then so are $\langle\Psi, \neg(\phi_2 \wedge \ldots \wedge \phi_n \wedge \phi_1)\rangle$ and $\langle\Psi, \neg(\phi_3 \wedge \cdots \wedge \phi_n \wedge \phi_1 \wedge \phi_2)\rangle$ so on. However, they are all identical (in the sense that each is more conservative than the others). We can ignore the unnecessary variants by just considering the canonical undercuts defined as follows.

Definition 3.4.2

An argument $\langle\Psi, \neg(\phi_1 \wedge \ldots \wedge \phi_n)\rangle$ is a **canonical undercut** for $\langle\Phi, \alpha\rangle$ iff it is a maximally conservative undercut for $\langle \Phi, \alpha \rangle$ and $\langle \phi_1, \ldots, \phi_n \rangle$ is the canonical enumeration of Φ.

Next is a simple and convenient characterization of the notion of a canonical undercut.

Proposition 3.4.2

An argument $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n) \rangle$ is a canonical undercut for $\langle \Phi, \alpha \rangle$ iff it is an undercut for $\langle \Phi, \alpha \rangle$ and $\langle \phi_1, \ldots, \phi_n \rangle$ is the canonical enumeration of Φ.

Corollary 3.4.1

A pair $\langle \Psi, \neg(\phi_1 \wedge \ldots \wedge \phi_n) \rangle$ is a canonical undercut for $\langle \Phi, \alpha \rangle$ iff it is an argument and $\langle \phi_1, \ldots, \phi_n \rangle$ is the canonical enumeration of Φ.

Clearly, an argument may have more than one canonical undercut. What do the canonical undercuts for the same argument look like? How do they differ from one another?

Proposition 3.4.3

Any two different canonical undercuts for the same argument have the same consequent but distinct supports.

Proposition 3.4.4

Given two different canonical undercuts for the same argument, neither is more conservative than the other.

Example 3.4.2

Let $\Delta = \{\alpha, \beta, \neg\alpha, \neg\beta\}$. Both of the following

$$\langle \{\neg\alpha\}, \neg(\alpha \wedge \beta) \rangle$$
$$\langle \{\neg\beta\}, \neg(\alpha \wedge \beta) \rangle$$

are canonical undercuts for $\langle \{\alpha, \beta\}, \alpha \leftrightarrow \beta \rangle$, but neither is more conservative than the other.

Proposition 3.4.5

For each defeater $\langle \Psi, \beta \rangle$ of an argument $\langle \Phi, \alpha \rangle$, there exists a canonical undercut for $\langle \Phi, \alpha \rangle$ that is more conservative than $\langle \Psi, \beta \rangle$.

That is, the set of all canonical undercuts of an argument represents all the defeaters of that argument (informally, all its counterarguments). This is to be taken advantage of in section 3.5.

(p.54) 3.4.1 Technical Developments

Restricting ourselves to maximally conservative undercuts forces us to check that they capture maximally conservative defeaters. In fact, they do, in a strong sense as given by proposition 3.4.6.

Proposition 3.4.6

If $\langle \Psi, \beta \rangle$ is a maximally conservative undercut for $\langle \Phi, \alpha \rangle$, then $\langle \Psi, \beta \rangle$ also is a maximally conservative defeater for $\langle \Phi, \alpha \rangle$.

As to a different matter, the converse of proposition 3.4.1 also holds.

Proposition 3.4.7

If $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n) \rangle$ is an undercut for $\langle \Phi, \alpha \rangle$ such that $\Phi = \{\phi_1, \ldots, \phi_n\}$, then it is a maximally conservative undercut for $\langle \Phi, \alpha \rangle$.

Corollary 3.4.2

Let $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n) \rangle$ be an undercut for $\langle \Phi, \alpha \rangle$. $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n) \rangle$ is a maximally conservative undercut for $\langle \Phi, \alpha \rangle$ iff $\Phi = \{\phi, \ldots, \phi_n\}$.

Although not strictly necessary, a sufficient condition for being a maximally conservative undercut is as follows.

Proposition 3.4.8

If $\langle \Psi, \beta \rangle$ is both a rebuttal and an undercut for $\langle \Phi, \alpha \rangle$ then $\langle \Psi, \beta \rangle$ is a maximally conservative undercut for $\langle \Phi, \alpha \rangle$.

In partial converse, there is a special case of the following general result.

Proposition 3.4.9

Let $\langle \Phi, \alpha \rangle$ be an argument. Each maximally conservative undercut for $\Phi, \alpha \rangle$ is a rebuttal for $\langle \Phi, \alpha \rangle$; iff Φ is logically equivalent with α.

In general, when an argument is a maximally conservative undercut for another argument, the converse does not hold but it is almost the case as shown next.

Proposition 3.4.10

If $\langle \Psi, \beta \rangle$ is a maximally conservative undercut for $\langle \Phi, \alpha \rangle$, then there exists $\Phi' \subseteq \Phi$ and γ such that $\langle \Phi', \gamma \rangle$ is a maximally conservative undercut for $\langle \Psi, \beta \rangle$.

Corollary 3.4.3

If $\langle \Psi, \beta \rangle$ is a canonical undercut for $\langle \Phi, \alpha \rangle$, then there exists $\Phi' \subseteq \Phi$ such that $\langle \Phi'$, $\neg(\psi_1 \wedge \cdots \wedge \psi_n) \rangle$ is a canonical undercut for $\langle \Psi, \beta \rangle$ (where $\langle \psi_1, \ldots, \psi_n$ is the canonical enumeration of Ψ).

Under certain circumstances, the notion of a rebuttal and the notion of maximally conservative coincide. One of these circumstances is given by the following result.

(p.55) Proposition 3.4.11

If $\langle \Phi, \alpha \rangle$ is an argument such that Φ is logically equivalent with α, then each rebuttal of $\langle \Phi, \alpha \rangle$ is equivalent with a canonical undercut for $\langle \Phi, \alpha \rangle$.

We will consider, in the next section, how canonical undercuts constitute a key concept for forming constellations of arguments and counterarguments.

3.5 Argument Trees

How does argumentation usually take place? Argumentation starts when an initial argument is put forward, making some claim. An objection is raised in the form of a counterargument. The latter is addressed in turn, eventually giving rise to a counter-counterargument, if any. And so on. However, there often is more than one counterargument to the initial argument, and if the counterargument actually raised in the first place had been different, the counter-counterargument would have been different, too, and similarly the counter-counter-counterargument, if any, and so on. Argumentation would have taken a possibly quite different course.

So do we find all the alternative courses that could take place from a given initial argument? And is it possible to represent them in a rational way, let alone to answer the most basic question of how do we make sure that no further counterargument can be expressed from the information available?

Answers are provided below through the notion of argument trees, but we first present an example to make things a little less abstract. Just a word of warning: The details of reasoning from support to consequent for each (counter-)argument do not really matter; they are made explicit in the example for the sake of completeness only.

Example 3.5.1

There are rumors about Ms. Shy expecting Mr. Scoundrel to propose to her, although she may not wish to get married. (To respect privacy, the names have been changed.) Here is the situation:

- • If he doesn't propose to her unless he finds she is rich, then he has no qualms.
- • If he proposes to her only if she looks sexy, then it is not the case that he has no qualms.
- • He finds out that she is rich.
- • She looks sexy.
- • He doesn't propose to her!

(p.56) There are grounds for various arguments about whether Mr. Scoundrel has no qualms about marrying Ms. Shy:

An argument claiming that Mr. Scoundrel has no qualms

As stated, if he doesn't propose to her unless he finds she is rich, then he has no qualms. In other words, if it is false that he has no qualms, it is false that he doesn't propose to her unless he finds she is rich. Equivalently, if it is false that he has no qualms, then he proposes to her while he does not find she is rich—which is not the case: He finds she is rich, and it is possible to conclude, by modus tollens, that he has no qualms.

A first counterargument

The sentence "at least in the event that he proposes to her, she looks sexy" is true: See the fourth statement. However, the sentence means the same as "he proposes to her only if she looks sexy." According to "if he proposes to her only if she looks sexy, then it is not the case that he has no qualms," it follows that it is not the case that he has no qualms.

A second counterargument

The claim in the initial argument is based on the condition "he proposes to her only if he finds she is rich" that can be challenged: Clearly, not both conclusions of "if he proposes to her only if he finds she is rich, then he has no qualms" and "if he proposes to her only if she looks sexy, then it is not the case that he has no qualms" are true. Hence, "he proposes to her only if he finds she is rich" is false if "he proposes to her only if she looks sexy" is true—which is the case as detailed in the first counterargument on the ground that she looks sexy.

A counter-counterargument

It is asserted that if he doesn't propose to her unless he finds she is rich, then he has no qualms. Stated otherwise, if it is false that he has no qualms, then it is false that he doesn't propose to her unless he finds she is rich. Equivalently, if it is false that he has no qualms, then he proposes to her while he does not find she is rich—which is not the case: He doesn't propose to her, and it is possible to conclude, by modus tollens, that he has no qualms.

Using the following propositional atoms

- p He proposes to her
- q He has qualms
- r He finds out that she is rich
- s She looks sexy

the statements are formalized as **(p.57)**

$(\neg r \rightarrow \neg p) \rightarrow \neg q$	If he doesn't propose to her unless he finds she is rich, then he has no qualms

$(p \to s) \to \neg\neg q$	If he proposes to her only if she looks sexy, then it is not the case that he has no qualms
r	He finds out that she is rich
s	She looks sexy
$\neg p$	He does not propose to her

and the arguments presented above take the form

$\langle\{r, (\neg r \to \neg p) \to \neg q\}, \neg q\rangle$	(initial argument I)
$\langle\{s, (p \to s) \to \neg\neg q\}, \neg\neg q\rangle$	(counterargument C_1)
$\langle\{s, (p \to s) \to \neg\neg q, (\neg r \to \neg p) \to \neg q\}, \neg(p \to r)\rangle$	(counter-counterargument C_2)
$\langle\{\neg p, (\neg r \to \neg p) \to \neg q\}, \neg q\rangle$	(counter-counterargument C)

There are still very many other counter^n-arguments (whatever n), despite the fact that definition 3.2.1 already rules out a number of informal ones.

Indeed, the argumentation about Mr. Scoundrel's qualms can take *different courses*. One is I, C_1, C, …, another is I, C_2, C, …, and there are many others issued from I. It would thus be useful to have an exhaustive account of the possible arguments and how they relate with respect to the initial argument (allowing us to reconstruct every possible course of argumentation starting with a given initial argument). And this is what argument trees are meant to do: An argument tree describes the various ways a given initial argument can be challenged, as well as how the counterarguments to the initial argument can themselves be challenged, and so on, recursively. However, some way of forming sequences of counterarguments is desirable, if not imperative, in view of corollary 3.3.2, which points out that argumentation often takes a course consisting of an infinite sequence of arguments, each being a counterargument to the preceding one.

Example 3.5.2

Argumentation sometimes falls on deaf ears, most often when simplistic arguments are uttered as illustrated below with a case of the "chicken and egg dilemma." Here the same arguments are used in a repetitive cycle:

Dairyman

Egg was first.

Farmer

Chicken was first.

Dairyman

Egg was first.

(p.58) *Farmer*

Chicken was first.

… …

The following propositional atoms are introduced:

- p Egg was first.
- q Chicken was first.
- r The chicken comes from the egg.
- s The egg comes from the chicken.

Admittedly, that the chicken was first and that the egg was first are not equivalent (i.e., $\neg(p \leftrightarrow q)$). Also, the egg comes from the chicken (i.e., r), and the chicken comes from the egg (i.e., s). Moreover, if the egg comes from the chicken, then the egg was not first. (i.e., $r \rightarrow \neg q$). Similarly, if the chicken comes from the egg, then the chicken was not first (i.e., $s \rightarrow \neg p$). Then, the above dispute can be represented as follows:

$$\begin{array}{c} \langle\{s \to \neg p, s, \neg(p \leftrightarrow q)\}, q\rangle \\ \uparrow \\ \langle\{r \to \neg q, r, \neg(p \leftrightarrow q)\}, p\rangle \\ \uparrow \\ \langle\{s \to \neg p, s, \neg(p \leftrightarrow q)\}, q\rangle \\ \uparrow \\ \langle\{r \to \neg q, r, \neg(p \leftrightarrow q)\}, p\rangle \\ \uparrow \\ \vdots \end{array}$$

We are now ready for our definition (below) of an argument tree in which the root of the tree is an argument of interest, and the children for any node are the canonical undercuts for that node. In the definition, we avoid the circularity seen in the above example by incorporating an intuitive constraint.

Definition 3.5.1

An **argument tree** for α is a tree where the nodes are arguments such that

1. The root is an argument for α.
2. For no node $\langle\Phi, \beta\rangle$ with ancestor nodes $\langle\Phi_1, \beta_1\rangle, \ldots, \langle\Phi_n, \beta_n\rangle$ is Φ a subset of $\Phi_1 \cup \cdots \cup \Phi_n$.

3. The children nodes of a node N consist of all canonical undercuts for N that obey 2.

(p.59) Note, for the definition of argument tree, in chapter 4 onwards, we assume a relaxed version of condition 3 in which “the children nodes of a node N consist of some or all of the canonical undercuts for N that obey 2.” In addition, for chapter 4 onwards, we use the term **complete argument tree** when we need to stress that condition 3 is “the children nodes of a node N consist of all the canonical undercuts for N that obey 2.”

We illustrate the definition of an argument tree in the following examples.

Example 3.5.3

Speaking of Simon Jones, once again …

- p Simon Jones is a Member of Parliament.
- $p \rightarrow \neg q$ If Simon Jones is a Member of Parliament, then we need not keep quiet about details of his private life.
- r Simon Jones just resigned from the House of Commons.
- $r \rightarrow \neg p$ If Simon Jones just resigned from the House of Commons, then he is not a Member of Parliament.
- $\neg p \rightarrow q$ If Simon Jones is not a Member of Parliament, then we need to keep quiet about details of his private life.

The situation can be depicted as follows:

$$\langle\{p, p \rightarrow \neg q\}, \neg q\rangle$$
$$\langle\{r, r \rightarrow \neg p\}, \neg p\rangle$$

Example 3.5.4

An obnoxious vice-president …

Vice-President

The only one not taking orders from me is the president; you are not the president but a regular employee; hence you take orders from me.

Secretary

I am the president’s secretary, not a regular employee.

Secretary

Anyway, I don't take orders from you.

This can be captured in terms of the sentences below (as recounted by the secretary).

- t I take orders from the vice-president.
- p I am the president.
- r I am a member of the regular staff.
- s I am the president's secretary.

The statements uttered are as follows:

(p.60)

- $\neg t \rightarrow p$ The only one not taking orders from me is the president.
- $\neg p \wedge r$ You are not the president but a regular employee.
- $s \wedge \neg r$ I am the president's secretary, not a regular employee.
- $\neg t$ I don't take orders from you.

We obtain the following argument tree:

$$\langle\{\neg t \rightarrow p, \neg p \wedge r\}, t\rangle$$
$$\nearrow \qquad \nwarrow$$
$$\langle\{s \wedge \neg r\}, \neg((\neg t \rightarrow p) \wedge (\neg p \wedge r))\rangle \qquad \langle\{\neg t\}, \neg((\neg t \rightarrow p) \wedge (\neg p \wedge r))\rangle$$

We give a further illustration of an argument tree in example 3.5.5, and then we motivate the conditions of definition 3.5.1 as follows: Condition 2 is meant to avoid the situation illustrated by example 3.5.6, and condition 3 is meant to avoid the situation illustrated by example 3.5.7.

Example 3.5.5

Given $\Delta = \{\alpha, \alpha \rightarrow \beta, \gamma, \gamma \rightarrow \neg\alpha, \neg\gamma \vee \neg\alpha\}$, we have the following argument tree:

$$\langle\{\alpha, \alpha \rightarrow \beta\}, \beta\rangle$$
$$\nearrow \qquad \nwarrow$$
$$\langle\{\gamma, \gamma \rightarrow \neg\alpha\}, \neg(\alpha \wedge (\alpha \rightarrow \beta))\rangle \qquad \langle\{\gamma, \neg\gamma \vee \neg\alpha\}, \neg(\alpha \wedge (\alpha \rightarrow \beta))\rangle$$

Note the two undercuts are equivalent. They do count as two arguments because they are based on two different items of the knowledgebase (even though these items turn out to be logically equivalent).

We adopt a lighter notation, writing $\langle\Psi, \diamond\rangle$ for a canonical undercut of $\langle\Phi, \beta\rangle$. Clearly, $\diamond$ is $\neg(\phi_1 \wedge \cdots \wedge \phi_n)$ where $\langle\phi_1, \ldots, \phi_n\rangle$ is the canonical enumeration for Φ.

Example 3.5.6

Let $\Delta = \{\alpha, \alpha \to \beta, \gamma \to \neg\alpha, \gamma\}$.

$$\begin{array}{c} \langle\{\alpha, \alpha \to \beta\}, \beta\rangle \\ \uparrow \\ \langle\{\gamma, \gamma \to \neg\alpha\}, \diamond\rangle \\ \uparrow \\ \langle\{\alpha, \gamma \to \neg\alpha\}, \diamond\rangle \end{array}$$

This is not an argument tree because condition 2 is not met. The undercut to the undercut is actually making exactly the same point (that α and γ are incompatible) as the undercut itself does, just by using modus tollens instead of modus ponens.

(p.61) Example 3.5.7

Given $\Delta = \{\alpha, \beta, \alpha \to \gamma, \beta \to \delta, \neg\alpha \vee \neg\beta\}$, consider the following tree:

$$\begin{array}{ccc} & \langle\{\alpha, \beta, \alpha \to \gamma, \beta \to \delta\}, \gamma \wedge \delta\rangle & \\ \nearrow & & \nwarrow \\ \langle\{\alpha, \neg\alpha \vee \neg\beta\}, \neg\beta\rangle & & \langle\{\beta, \neg\alpha \vee \neg\beta\}, \neg\alpha\rangle \end{array}$$

This is not an argument tree because the two children nodes are not maximally conservative undercuts. The first undercut is essentially the same argument as the second undercut in a rearranged form (relying on α and β being incompatible, assume one and then conclude that the other doesn't hold). If we replace these by the maximally conservative undercut $\langle\{\neg\alpha \vee \neg\beta\}, \diamond\rangle$, we obtain an argument tree.

The following result is important in practice—particularly in light of corollary 3.3.2 and also other results we present in section 3.6.

Proposition 3.5.1

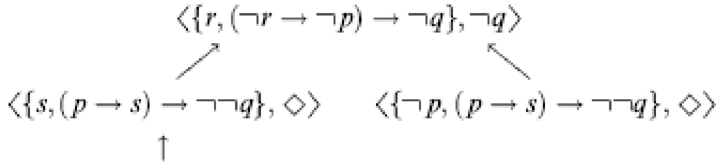
Argument trees are finite.

The form of an argument tree is not arbitrary. It summarizes all possible courses of argumentation about the argument in the root node. Each node except the root node is the starting point of an implicit series of related arguments. What happens is that for each possible course of argumentation (from the root), an initial sequence is provided as a branch of the tree up to the point that no subsequent counter^{n}-argument needs a new item in its support (where "new" means not occurring somewhere in that initial sequence). Also, the counterarguments in a course of argumentation may somewhat differ from the ones in the corresponding branch of the argument tree.

Example 3.5.8

The statements about Ms. Shy and Mr. Scoundrel can be captured in $\Delta = \{(\neg r \to$

$\neg p) \rightarrow \neg q$, $(p \rightarrow s) \rightarrow \neg\neg q$, r, s, $\neg p\}$. The argument tree with $\langle \{r, (\neg r \rightarrow \neg p) \rightarrow \neg q\},$



$\neg q\rangle$ as its root is $\langle \{\neg p, (\neg r \rightarrow \neg p) \rightarrow \neg q\}, \Diamond \rangle$

The left-most branch clearly captures any courses I, C_1, C, … Note that the next element (after C) could be C_2, but that is accounted for because the support of C_2 is a subset of the set-theoretic union of the supports of I, C_1, and C (actually, the resulting set-theoretic union gives Δ). Less **(p.62)** immediate is the fact that the left-most branch also captures any courses I, C_2, C, … The key fact is that the support of C_1 is a subset of the support of C_2. What about any courses I, C_1, I, … ? These are captured through the fact that C_1 being a canonical undercut of I, it happens that appropriately changing just the consequent in I gives a canonical undercut of C_1 (cf. corollary 3.4.3). That is the idea that each node (except for the root) is the starting point of an implicit series of related arguments. Lastly, the right-most branch captures courses that were not touched upon in the earlier discussion of the example.

Example 3.5.9

Let us return to the "chicken and egg dilemma":

Dairyman

Egg was first.

Farmer

Chicken was first.

Dairyman

Egg was first.

Farmer

Chicken was first.

… …

Here are the formulae again:

- p Egg was first.
- q Chicken was first.

- r The chicken comes from the egg.
- s The egg comes from the chicken.
- $\neg(p \leftrightarrow q)$ That the egg was first and that the chicken was first are not equivalent.
- $r \rightarrow \neg q$ The chicken comes from the egg implies that the chicken was not first.
- $s \rightarrow \neg p$ The egg comes from the chicken implies that the egg was not first.

Thus, $\Delta = \{\neg(p \leftrightarrow q), r \rightarrow \neg q, s \rightarrow \neg p, r, s\}$. The argument tree with the dairyman's argument as its root is

$$\begin{array}{c}\langle\{r \rightarrow \neg q, r, \neg(q \leftrightarrow p)\}, p\rangle \\ \uparrow \\ \big\langle\big\{s \rightarrow \neg p, s, \neg(q \leftrightarrow p)\big\}, \diamond\big\rangle\end{array}$$

but it does **not** mean that the farmer has the last word nor that the farmer wins the dispute! The argument tree is merely a **representation** of the argumentation (in which the dairyman provides the initial argument).

(p.63) Although the argument tree is finite, the argumentation here is infinite and unresolved.

3.5.1 Some Useful Subsidiary Definitions

Here we provide a few further definitions that will be useful in the following chapters.

For an argument tree T, Depth(T) is the length of the longest branch of T, and Width(T) is the number of leaf nodes in T (see Appendix B for further details on trees). For an argument tree T, Nodes(T) is the set of nodes (i.e., arguments) in T and Root(T) is the root of T. We call the argument at the root of an argument tree T, i.e., Root(T), the **root argument** (or, equivalently, **initiating argument**). For an argument tree T, Subject(T) is Claim(Root(T)), and if Subject(T) is α, we call α the **subject** of T. Given an argument tree T, for an argument A, Parent(A) is the parent of A in T (i.e., parenthood is defined in the context of a particular argument tree).

For an argument tree T, and an argument A, Siblings(T, A) is the set of siblings of A in T (i.e., it is the set of children of Parent(A)) and Undercuts(T, A) is the set of children of A. For an argument tree T, Siblings(T) is the set of sibling sets in T (i.e., $S \in$ Siblings(T) iff S = Undercuts(T, A) for some A in T).

For an argument tree T, each argument in T is either an **attacking argument** or a **defending argument**. If A_r is the root, then A_r is a defending argument. If an argument A_i is a defending argument, then any A_j whose parent is A_i is an attacking argument. If an argument A_j is an attacking argument, then any A_k whose parent is A_j is a defending argument. For an argument tree T, Defenders(T) is the set of defending arguments in T, and Attackers(T) is the set of attacking arguments in T.

3.5.2 Technical Developments

The question investigated here is as follows: Looking at the form of argument trees, what can be assessed about Δ?

Inconsistency for Δ was related, as stated in proposition 3.3.13 and corollary 3.3.5, to the existence of a defeater. This can be turned into a relationship between inconsistency for Δ and the case that argument trees consist of a single node as shown in the next result.

Proposition 3.5.2

If Δ is consistent, then all argument trees have exactly one node. The converse is true when each formula in Δ is consistent.

Conversely, looking at some properties about Δ, what can be said about argument trees?

(p.64) Proposition 3.5.3

Let T be an argument tree whose root node $\langle \Phi, \alpha \rangle$ is such that no subset of Δ is logically equivalent with α. Then, no node in T is a rebuttal for the root.

Example 3.5.10

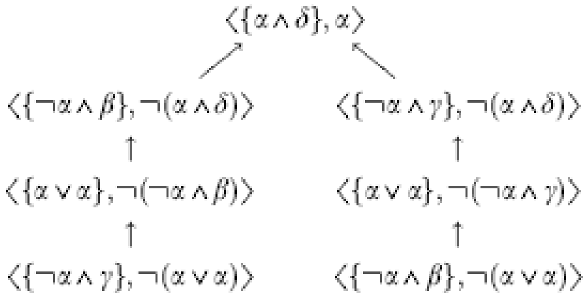
$\Delta = \{\alpha \wedge \beta \wedge \gamma, \delta \wedge \neg(\alpha \wedge \beta), \neg\alpha \wedge \neg\delta, \delta \rightarrow (\beta \wedge \gamma), \alpha \vee \alpha\}$.

$$\begin{array}{c}
\langle \{\alpha \wedge \beta \wedge \gamma\}, \alpha \rangle \\
\uparrow \\
\langle \{\delta \wedge \neg(\alpha \wedge \beta)\}, \neg(\alpha \wedge \beta \wedge \gamma) \rangle \\
\uparrow \\
\langle \{\neg\alpha \wedge \neg\delta\}, \neg(\delta \wedge \neg(\alpha \wedge \beta)) \rangle \\
\uparrow \\
\langle \{\alpha \vee \alpha\}, \neg(\neg\alpha \wedge \neg\delta) \rangle \\
\uparrow \\
\langle \{\delta \wedge \neg(\alpha \wedge \beta), \delta \rightarrow (\beta \wedge \gamma)\}, \neg(\alpha \vee \alpha) \rangle
\end{array}$$

The above argument tree contains a rebuttal of the root that is not the child of the root.

Example 3.5.11

$\Delta = \{\neg\alpha \wedge \beta, \neg\alpha \wedge \gamma, \alpha \wedge \delta, \alpha \vee \alpha\}$.



The above argument tree contains rebuttals of the root that are not children of the root.

3.6 Duplicates

Equivalent arguments are arguments that express the same reason for the same point. For undercuts, a more refined notion than equivalent arguments is useful:

Definition 3.6.1

Two undercuts $\langle \Gamma \cup \Phi, \neg\psi \rangle$ and $\langle \Gamma \cup \Psi, \neg\phi \rangle$ are **duplicates** of each other iff ϕ is $\phi_1 \wedge \cdots \wedge \phi_n$ such that $\Phi = \{\phi_1, \ldots, \phi_n\}$ and ψ is $\psi_1 \wedge \cdots \wedge \psi_m$ such that $\Psi = \{\psi_1, \ldots, \psi_m\}$.

(p.65) Duplicates introduce a symmetric relation that fails to be transitive (and reflexive). Arguments that are duplicates of each other are essentially the same argument in a rearranged form.

Example 3.6.1

The two arguments below are duplicates of each other:

$$\langle \{\alpha, \neg\alpha \vee \neg\beta\}, \neg\beta \rangle$$
$$\langle \{\beta, \neg\alpha \vee \neg\beta\}, \neg\alpha \rangle$$

Example 3.6.2

To illustrate the lack of transitivity in the duplicate relationship, the following two arguments are duplicates

$$\langle \{\gamma, \alpha, \alpha \wedge \gamma \rightarrow \neg\beta\}, \neg\beta \rangle$$
$$\langle \{\gamma, \beta, \alpha \wedge \gamma \rightarrow \neg\beta\}, \neg\alpha \rangle$$

and similarly the following two arguments are duplicates

$$\langle \{\gamma, \beta, \alpha \wedge \gamma \rightarrow \neg\beta\}, \neg\alpha \rangle$$
$$\langle \{\alpha, \alpha \wedge \gamma \rightarrow \neg\beta\}, \neg(\beta \wedge \gamma) \rangle$$

but the following two are not duplicates:

$$\langle\{\alpha, \alpha \wedge \gamma \rightarrow \neg\beta\}, \neg(\beta \wedge \gamma)\rangle$$
$$\langle\{\gamma, \alpha, \alpha \wedge \gamma \rightarrow \neg\beta\}, \neg\beta\rangle$$

The following proposition shows how we can systematically obtain duplicates. In this result, we see there is an explosion of duplicates for each maximally conservative undercut. This obviously is a potential concern for collating counterarguments.

Proposition 3.6.1

For every maximally conservative undercut $\langle\Psi, \beta\rangle$ to an argument $\langle\Phi, \alpha\rangle$, there exist at least $2^m - 1$ arguments, each of which undercuts the undercut (m is the size of Ψ). Each of these $2^m - 1$ arguments is a duplicate to the undercut.

Proposition 3.6.2

No two maximally conservative undercuts of the same argument are duplicates.

Corollary 3.6.1

No two canonical undercuts of the same argument are duplicates.

Proposition 3.6.3

No branch in an argument tree contains duplicates, except possibly for the child of the root to be a duplicate to the root.

These last two results are important. They show that argument trees are an efficient and lucid way of representing the pertinent counterarguments **(p.66)** to each argument: Corollary 3.6.1 shows it regarding breadth, and proposition 3.6.3 shows it regarding depth. Moreover, they show that the intuitive need to eliminate duplicates from argument trees is taken care of through an efficient syntactical criterion (condition 2 of definition 3.5.1).

3.7 Argument Structures

We now consider how we can gather argument trees for and against a claim. To do this, we define argument structures.

Definition 3.7.1

An **argument structure** for a formula α is a pair of sets $\langle P, C\rangle$, where P is the set of argument trees for α and C is the set of argument trees for $\neg\alpha$.

Example 3.7.1

Let $\Delta = \{\alpha \vee \beta, \alpha \rightarrow \gamma, \neg\gamma, \neg\beta, \delta \leftrightarrow \beta\}$. For this, we obtain three argument trees for

$$\langle\{\alpha\vee\beta,\neg\beta\},\alpha\vee\neg\delta\rangle \qquad \langle\{\delta\leftrightarrow\beta,\neg\beta\},\alpha\vee\neg\delta\rangle$$
$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$
$$\langle\{\alpha\rightarrow\gamma,\neg\gamma\},\diamond\rangle \qquad \langle\{\alpha\vee\beta,\alpha\rightarrow\gamma,\neg\gamma\},\diamond\rangle$$

$$\langle\{\alpha\vee\beta,\alpha\rightarrow\gamma,\neg\gamma,\delta\leftrightarrow\beta\},\neg\alpha\wedge\delta\rangle$$
$$\uparrow$$
$$\langle\{\neg\beta\},\diamond\rangle$$

the argument structure for a $\alpha \vee \neg\delta$.

Example 3.7.2

Let $\Delta = \{\alpha \leftrightarrow \beta, \beta \vee \gamma, \gamma \rightarrow \beta, \neg\alpha \vee \neg\beta \vee \neg\gamma, \gamma \wedge \delta, \neg\delta\}$. From this, we obtain the following argument trees for and against $\beta \wedge \delta$, where $\Phi_1 = \{\alpha \leftrightarrow \beta, \neg\alpha \vee \neg\beta \vee \neg\gamma, \gamma \wedge \delta\}$ and $\Phi_2 = \{\beta \vee \gamma, \gamma \rightarrow \beta, \alpha \leftrightarrow \beta, \neg\alpha \vee \neg\beta \vee \neg\gamma\}$.

$$\langle\{\gamma\wedge\delta,\gamma\rightarrow\beta\},\beta\wedge\delta\rangle$$
$$\nearrow \qquad \uparrow \qquad \nwarrow$$
$$\langle\Phi_1,\diamond\rangle \qquad \langle\{\neg\delta\},\diamond\rangle \qquad \langle\Phi_2,\diamond\rangle$$
$$\uparrow$$
$$\langle\{\neg\delta\},\diamond\rangle$$

$$\langle\{\alpha\leftrightarrow\beta,\neg\alpha\vee\neg\beta\vee\neg\gamma,\gamma\wedge\delta\},\neg(\beta\wedge\delta)\rangle$$
$$\nearrow \qquad\qquad \nwarrow$$
$$\langle\{\gamma\rightarrow\beta\},\diamond\rangle \qquad\qquad \langle\{\neg\delta\},\diamond\rangle$$

$$\langle\{\neg\delta\},\neg(\beta\wedge\delta)\rangle$$
$$\uparrow$$
$$\langle\{\gamma\wedge\delta\},\diamond\rangle$$

(p.67) Proposition 3.7.1

Let $\langle P, C\rangle$ be an argument structure. If there exists an argument tree in P that has exactly one node, then C is the empty set. The converse is untrue, even when assuming that P is nonempty.

Example 3.7.3

Let $\Delta = \{\alpha \vee \neg\beta, \beta, \neg\beta\}$. In the argument structure $\langle P, C\rangle$ for α, we have that C is the empty set while P contains an argument tree that has more than one node:

$$\langle\{\alpha\vee\neg\beta,\beta\},\alpha\rangle$$
$$\uparrow$$
$$\langle\{\neg\beta\},\diamond\rangle$$

Example 3.7.3 illustrates the last sentence in proposition 3.7.1. If Δ is augmented with $\alpha \wedge \beta$, for instance, then $\langle P, C\rangle$ is such that P contains both an argument tree with more than one node and an argument tree consisting of just a root node.

Proposition 3.7.2

Let $\langle P, C \rangle$ be an argument structure. If Δ is consistent, then each argument tree in P has exactly one node and C is the empty set. The converse is untrue, even when assuming that P is nonempty and that each formula in Δ is consistent.

The last sentence in the statement of proposition 3.7.2 can be illustrated by the following counterexample.

Example 3.7.4

Let $\Delta = \{\alpha, \beta, \neg\beta\}$. The argument structure $\langle P, C \rangle$ for α is such that P contains a single argument tree consisting of just the root node below:

$$\langle \{\alpha\}, \alpha \rangle$$

In argument structures, P and C are symmetrical. Any property enjoyed by one has a counterpart, which is a property enjoyed by the other: Both are the same property, with P and C exchanged. For example, we have the result similar to proposition 3.7.1 stating that if there exists an argument tree in C that has exactly one node, then P is the empty set. Symmetry goes even deeper, inside the argument trees of P and C. This is exemplified in the next result.

Proposition 3.7.3

Let $\langle [X_1, \ldots, X_n], [Y_1, \ldots, Y_m] \rangle$ be an argument structure. For any i and any j, the support of the root node of Y_j (resp. X_i) is a superset of the support of a canonical undercut for the root node of X_i (resp. Y_j).

Proposition 3.7.3 is reminiscent of the phenomenon reported in corollary 3.3.1.

(p.68) Proposition 3.7.4

Let $\langle P, C \rangle$ be an argument structure. Then, both P and C are finite.

3.8 First-Order Argumentation

In many situations, it is apparent that there is a need to support first-order argumentation. As an example, consider a senior clinician in a hospital who may need to consider the pros and cons of a new drug regime in order to decide whether to incorporate the regime as part of hospital policy: This could be expedited by considering the pros and cons of a first-order statement formalizing that piece of policy.

As another example, consider an information systems consultant who is collating requirements from users within an organization. Due to conflicts between requirements from different users, the consultant may need to consider arguments for and against particular requirements being adopted in the final requirements specification. Towards this end, first-order statements provide a format for readily and thoroughly capturing constraints and compromises.

It is important to notice that using a propositional framework to encode first-order statements leads to mishaps, for example, when attempting to mimic $\forall x.\alpha[x]$ by means of its instances $\alpha[t]$ for all ground elements t in the universe of discourse: Due to circumstantial properties, it may happen that, whatever t, a particular argument for $\alpha[t]$ can be found but there is no guarantee that an argument for $\forall x.\alpha[x]$ exists. Here is an example. Consider the statements "if x satisfies p and q, then x satisfies r or s" and "if x satisfies q and r and s, then x satisfies t." Clearly, these do not entail the statement "if x satisfies p, then x satisfies t." Assume the set of all ground terms from the knowledgebase is $\{a, b\}$. The obvious idea is to consider $\forall x.\alpha[x]$ as being equivalent with both instances $\alpha[a]$ and $\alpha[b]$. Unfortunately, should $q(a)$ and $r(a) \vee s(a) \rightarrow t(a)$ be incidentally the case as well as $s(b)$ and $p(b) \rightarrow q(b) \wedge r(b)$, then "if x satisfies p then x satisfies t" would be regarded as argued for! The moral is that a propositional approach here cannot be substituted for a first-order one. In such situations, a first-order approach cannot be dispensed with.

To address this need for first-order argumentation, we generalize our proposal from the propositional case to the first-order case. For a first-order language $\mathcal{C}$, the set of formulae that can be formed is given by the usual inductive definitions for classical logic: Roman symbols $p, q, \ldots$ denote predicates, Greek symbols $\alpha, \beta, \ldots$ denote formulae.

All the definitions for argument, counterargument, rebuttal, undercut, maximally conservative undercut, canonical undercut, and argument tree **(p.69)** are the same as for the propositional case, except that we assume Δ is a set of first-order formulae and that $\vdash$ is the first-order consequence relation. Given that this migration from the propositional case to the first-order case is straightforward, we do not repeat any of the definitions, but instead we just provide some examples to illustrate the use of these definitions in the first-order case.

Example 3.8.1

Consider the following knowledgebase:

$$\Delta = \{\forall x.\,(p(x) \rightarrow q(x) \vee r(r)), p(a), \neg\forall x.\,s(x), \neg\exists x.\,r(x),$$
$$\neg\exists x.\,(p(x) \rightarrow q(x) \vee r(x))\}$$

Some arguments from the knowledgebase are listed below:

$$\langle\{p(a), \forall x.\,(p(x) \rightarrow q(x) \vee r(x))\}, q(a) \vee r(a)\rangle$$
$$\langle\{\neg\forall x.\,s(x)\}, \neg\forall x.\,s(x)\rangle$$
$$\langle\{\neg\exists x.\,r(x)\}, \forall x.\,\neg r(x)\rangle$$

Example 3.8.2

Given Δ as in example 3.8.1, the first argument (below) is a more conservative argument than the second:

$$\langle\{p(a), \forall x.(p(x) \rightarrow q(x) \vee r(x))\}, q(a) \vee r(a)\rangle$$
$$\langle\{p(a), \forall x.(p(x) \rightarrow q(x) \vee r(x)), \neg\exists x.r(x)\}, q(a)\rangle$$

Example 3.8.3

Again, Δ is as in example 3.8.1. It is easy to find an undercut for the argument $\langle\{p(a), \forall x.(p(x) \rightarrow q(x) \vee r(x))\}, q(a) \vee r(a)\rangle$; an obvious one is $\langle\{\neg\exists x.(p(x) \rightarrow q(x) \vee r(x))\}, \neg\forall x.(p(x) \rightarrow q(x) \vee r(x))\rangle$. Now, there is another one, which actually is more conservative: $\langle\{\neg\exists x.(p(x) \rightarrow q(x) \vee r(x))\}, \neg(p(a) \wedge \forall x.(p(x) \rightarrow q(x) \vee r(x)))\rangle$.

Example 3.8.4

Given an appropriate Δ and provided the conditions for definition 3.2.1 are met, we have the general cases below:

$$\langle\{\forall x.\alpha[x]\}, \alpha[a]\rangle \text{ is undercut by } \langle\{\neg\exists x.\alpha[x]\}, \neg\forall x.\alpha[x]\rangle$$
$$\langle\{\forall x.\alpha[x]\}, \alpha[a]\rangle \text{ is undercut by } \langle\{\exists x.\neg\alpha[x]\}, \neg\forall x.\alpha[x]\rangle$$
$$\langle\{\forall x.\alpha[x]\}, \alpha[a]\rangle \text{ is undercut by } \langle\{\neg\alpha[b]\}, \neg\forall x.\alpha[x]\rangle$$
$$\langle\{\forall x.\alpha[x]\}, \alpha[a]\rangle \text{ is undercut by } \langle\{\neg\alpha[c]\}, \neg\forall x.\alpha[x]\rangle$$

Example 3.8.5

If Δ is as in example 3.8.1, a maximally conservative undercut for the first argument below is the second argument.

$$\langle\{p(a), \forall x.(p(x) \rightarrow q(x) \vee r(x))\}, q(a) \vee r(a)\rangle$$
$$\langle\{\neg\exists x.(p(x) \rightarrow q(x) \vee r(x))\}, \neg(p(a) \wedge \forall x.(p(x) \rightarrow q(x) \vee r(x)))\rangle$$

(p.70) Example 3.8.6

Let Δ be as in example 3.8.1. A complete argument tree for $q(a)$ is as follows:

$$\langle\{p(a), \forall x.(p(x) \rightarrow q(x) \vee r(x)), \neg\exists x.r(x)\}, q(a)\rangle$$
$$\uparrow$$
$$\langle\{\neg\exists x.(p(x) \rightarrow q(x) \vee r(x))\}, \diamond\rangle$$

Example 3.8.7

Let $\Delta = \{\forall x.(p(x) \wedge q(x)), \exists x.\neg(p(x) \wedge q(x)), \forall x.(r(x) \wedge \neg p(x)), \forall x \forall y.s(x, y), \forall x.(s(x, f(x)) \rightarrow \neg r(x))\}$. An argument tree for $\forall x.q(x)$ is as follows:

$$\langle\{\forall x.(p(x) \wedge q(x))\}, \forall x.q(x)\rangle$$

$$\nearrow \qquad\qquad \nwarrow$$

$$\langle\{\exists x.\neg(p(x) \wedge q(x))\}, \diamond\rangle \qquad \langle\{\forall x.(r(x) \wedge \neg p(x))\}, \diamond\rangle$$

$$\uparrow$$

$$\langle\{\forall x \forall y.s(x,y), \forall x.(s(x, f(x)) \rightarrow \neg r(x))\}, \diamond\rangle$$

Thus, as in the propositional case, an argument tree is an efficient representation of the counterarguments, counter-counterarguments, …, in the first-order case.

Proposition 3.8.1

Let $\alpha \in \mathcal{L}$. If Δ is finite, there are a finite number of argument trees with the root being an argument with consequent α that can be formed from Δ, and each of these trees has finite branching and a finite depth.

From these examples, and result, we see that we can straightforwardly use our framework for the first-order case. All the results we have presented for the propositional can be immediately generalized to the first-order case.

We finish this section with a larger example of first-order reasoning.

Example 3.8.8

Consider the following literals concerning a country called Landia:

*n*1 ReportDate(Landia, 30May2000)
*n*2 Government(Landia, unstable)
*n*3 Democracy(Landia, strong)
*n*4 PublicSpending(Landia, excessive)
*n*5 OilExport(Landia, significant)
*n*6 OilPrice(Landia, increasing)
*n*7 LastElection(Landia, recent)
*n*8 Currency(Landia, strong)

(p.71) Consider also the following general knowledge about countries:
*d*1 ∀X.Government(X, unstable) → credit-risk(X)
*d*2 ∀X.Democracy(X, strong) ∧ LastElection(X, recent) → ¬Government(X, unstable)
*d*3 ∀X.PublicSpending(X, excessive) → credit–risk(X)
*d*4 ∀X.OilExport(X, significant) ∧ OilPrice(X, increasing) → ¬PublicSpending(X, excessive)
*d*5 ∀X.Currency(X, strong) → ¬credit-risk(X)

Now assume the knowledgebase to be $\Delta = \{n_1, \ldots, n8, d1, \ldots, d5\}$. Note, for a more lucid presentation, we use the labels for the formulae (rather than the formulae themselves) in the supports of the arguments. From this, we obtain the following argument trees for and against the inference credit-risk(Landia).

$\langle\{n2, d1\}, \texttt{credit-risk(Landia)}\rangle$

$\nearrow \qquad \nwarrow$

$\langle\{n3, n7, d2\}, \diamond\rangle \qquad \langle\{n8, d5\}, \diamond\rangle$

$\uparrow$

$\langle\{n4, d3\}, \diamond\rangle$

$\uparrow$

$\langle\{n5, n6, d4\}, \diamond\rangle$

$\langle\{n4, d3\}, \texttt{credit-risk(Landia)}\rangle$

$\nearrow \qquad \nwarrow$

$\langle\{n5, n6, d4\}, \diamond\rangle \qquad \langle\{n8, d5\}, \diamond\rangle$

$\uparrow$

$\langle\{n2, d1\}, \diamond\rangle$

$\uparrow$

$\langle\{n3, n7, d2\}, \diamond\rangle$

$\langle\{n8, d5\}, \neg\texttt{credit-risk(Landia)}\rangle$

$\nearrow \qquad \nwarrow$

$\langle\{n2, d1\}, \diamond\rangle \qquad \langle\{n4, d3\}, \diamond\rangle$

$\uparrow \qquad \uparrow$

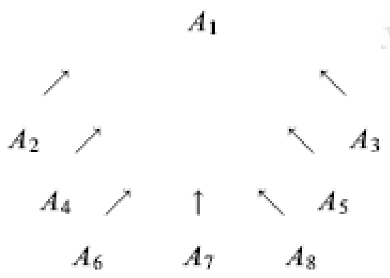
$\langle\{n3, n7, d2\}, \diamond\rangle \qquad \langle\{n5, n6, d4\}, \diamond\rangle$

(p.72) Given the simple nature of the knowledgebase in the above example, we see that each tree is a different arrangement of the same set of arguments. However, we stress that each tree is a stand-alone representation of the set of arguments from a particular perspective and so it is necessary to have all these trees.

3.9 Discussion

In this chapter, we have proposed a framework for modeling argumentation. The key features of this framework are the clarification of the nature of arguments and counterarguments; the identification of canonical undercuts, which we argue are the only undercuts that we need to take into account; and the representation of argument trees and argument structures that provide a way of exhaustively collating arguments and counterarguments.

Distinct but logically equivalent supports give rise to different canonical undercuts that all have to occur in a complete argument tree whose root node is attacked by these supports. The reader may wonder what the rationale is here, as in the following case, where A_2, … , A_8 are canonical undercuts of A_1:



where

$$\begin{aligned}
A_1 &= \langle\{\alpha \vee \beta\}, \alpha \vee \beta\rangle \\
A_2 &= \langle\{\neg\alpha \wedge \neg\beta\}, \diamond\rangle \\
A_3 &= \langle\{\neg\beta \wedge \neg\alpha\}, \diamond\rangle \\
A_4 &= \langle\{\neg(\beta \leftrightarrow \neg\alpha), \alpha \wedge \beta \rightarrow \neg\alpha \vee \neg\beta\}\diamond\rangle \\
A_5 &= \langle\{\neg\alpha \vee \neg\beta, \alpha \leftrightarrow \beta\}, \diamond\rangle \\
A_6 &= \langle\{\alpha \vee \beta \rightarrow \neg\alpha \wedge \neg\beta\}, \diamond\rangle \\
A_7 &= \langle\{\neg\alpha, \neg\beta\}, \diamond\rangle \\
A_8 &= \langle\{\neg(\alpha \vee \beta)\}, \diamond\rangle
\end{aligned}$$

(p.73) There are so many (infinitely many, in fact) *logically equivalent* ways to refute $\alpha \vee \beta$, so why spell out many counterarguments that are so close to one another? Here is an explanation. An argument tree is intended to be exhaustive in recording the ways the argument can *actually* be challenged, but that does not mean that the argument tree lists all the ways suggested by classical logic: Remember that an argument must have a subset of Δ as its support. And Δ is *not* closed under logical equivalence! Thus, only those logically equivalent supports that are *explicitly* mentioned by means of Δ give rise to arguments to be included in an argument tree. That makes a big difference, and it is where the rationale stands: If logically equivalent forms have been explicitly provided, it must be for some reason. For example, the above tree is not to be an argument tree unless there were some good reason to have all these many variants of $\{\neg\alpha, \neg\beta\}$ in Δ.

If the argument tree is used for presenting the arguments and counterarguments to a user, then the user would only want to see those arguments that have a good reason to be there. Redundant arguments are unlikely to be welcomed by the user. Should an argument tree eventually to be used when doing some kind of evaluation based on reinforcement, distinct albeit logically equivalent evidence may prove useful (e.g., if a surgeon has two distinct arguments for undertaking a risky operation, these arguments would reinforce each other), whereas statements that merely happen to be logically equivalent should not be included in the constellation.

Superficially, an argument structure could be viewed as an argument framework in Dung's system. An argument in an argument tree could be viewed as an argument in a

Dung argument framework, and each arc in an argument tree could be viewed as an attack relation. However, the way sets of arguments are compared is different.

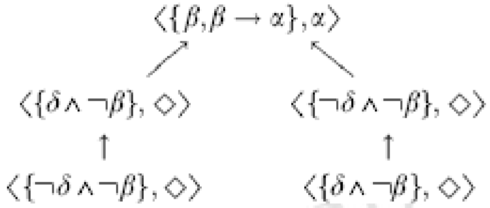
Some differences between Dung's approach and our approach can be seen in the following examples.

Example 3.9.1

Consider a set of arguments $\{a_1, a_2, a_3, a_4\}$ with the attack relation $\mathcal{R}$ such that $a_2\mathcal{R}a_1$, $a_3\mathcal{R}a_2$, $a_4\mathcal{R}a_3$, and $a_1\mathcal{R}a_4$. Here there is an admissible set $\{a_1, a_3\}$. We can try to construct an argument tree with a_1 at the root. As a counterpart to the attack relation, we regard that a_1 is undercut by a_2, a_2 is undercut by a_3, and so on. However, the corresponding sequence of nodes a_1, a_2, a_3, a_4, a_1 is not an argument tree because a_1 occurs twice in the branch (violating condition 2 of definition 3.5.1). Thus, the form of the argument tree for a_1 fails to represent the fact that a_1 attacks a_4.

(p.74) Example 3.9.2

Let $\Delta = \{\beta, \beta \to \alpha, \delta \wedge \neg\beta, \neg\delta \wedge \neg\beta\}$, giving the following argument tree for α:



For this, let a_1 be $\langle\{\beta, \beta \to \alpha\}, \alpha\rangle$, a_2 be $\langle\{\delta \wedge \neg\beta\}, \diamond\rangle$ and a_3 be $\langle\{\neg\delta \wedge \neg\beta\}, \diamond\rangle$. Disregarding the difference between the occurrences of $\diamond$, this argument tree rewrites as $a_2\mathcal{R}a_1$, $a_3\mathcal{R}a_1$, $a_3\mathcal{R}a_2$, and $a_2\mathcal{R}a_3$, where a_1 denotes the root node $\langle\{\beta, \beta \to \alpha\}, \alpha\rangle$. In this argument tree, each defeater of the root node is defeated. Yet no admissible set of arguments contains a_1.

Finally, we can consider capturing a class of arguments that fail to be deductive. In other words, we can revisit one of the basic assumptions made at the start of this chapter. For this, the basic principle for our approach still applies: An argument comes with a claim, which relies on reasons by virtue of some given relationship between the reasons and the claim. Thus, arguments can still be represented by pairs, but the relationship is no longer entailment in classical logic; it is a binary relation of some kind capturing "tentative proofs" or "proofs using nonstandard modes of inference" instead of logical proofs.

This relationship can be taken to be almost whatever pleases you provided that you have a notion of consistency. Observe that this does not mean that you need any second element of a pair to stand for "absurdity": You simply have to specify a subset of the pairs to form the cases of inconsistency. Similarly, our approach is not necessarily restricted to a logical language, and another mode of representation can be chosen.

3.10 Bibliographic Notes

This chapter is based on a particular proposal for logic-based argumentation for the propositional case [BH00, BH01] and generalized to the first-order case [BH05]. We have compared this proposal with abstract argumentation and shown that by having a logic-based notion of argumentation, we can have a much deeper understanding of the individual arguments and of the counterarguments that impact on each argument. Furthermore, by introducing logical knowledgebases, we can automatically **(p.75)** construct individual arguments and constellations of arguments and counterarguments.

This proposal is not the first proposal for logic-based argumentation. However, most proposals are not based on classical logic. Since we believe that understanding argumentation in terms of classical logic is an ideal starting point for understanding the elements of argumentation, we have focused on classical logic.

In comparison with the previous proposals based on classical logic (e.g., [AC98, Pol92]), our proposal provides a much more detailed analysis of counterarguments, and ours is the first proposal to consider canonical undercuts. Canonical undercuts are a particularly important proposal for ensuring that all the relevant undercuts for an argument are presented, thereby ensuring that a constellation of arguments and counterarguments is exhaustive, and yet ensuring that redundant arguments are avoided from this presentation.

We leave a more detailed comparison with other proposals for logic-based argumentation (both proposals based on classical logic and proposals based on defeasible logic) until chapter 8. **(p.76)**